HILBERT-BASED VORTICITY STATISTICS IN TWO-DIMENSIONAL TURBULENCE

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Abstract In this paper, we report for a measurement of the high-order scaling exponents $\zeta_\omega^F(q)$ (forward) and $\zeta_\omega^I(q)$ (inverse) for the vorticity in two-dimensional (2D) turbulence. This is accomplished by applying a Hilbert-based technique to the vorticity fields obtained from a 2D high-resolution direct numerical simulation with an Ekman friction. It confirms that the inverse enstrophy cascade is nonintermittent with a scaling exponent linear with the statistical order q, i.e., $q/3$. While the measured scaling exponents $\zeta_\omega^F(q)$ is found to be nonlinear with q and can be described by a log-Poisson like formula, i.e., $\zeta_\omega^F(q) = q/3 + 2/5(1-(2/5)^q)$. This confirms experimentally that the forward enstrophy cascade in the 2D turbulence is intermittent.

INTRODUCTION

Two dimensional (2D) turbulence is an ideal model for several turbulent flows, such as the first approximation to the large-scale motion in atmosphere and oceans, etc.,[2, 3]. The 2D turbulence and relative problems have attracted a lot of attentions in recent years. Specifically for the small scale motions, it is believed that there exists a dual cascades, i.e., a forward enstrophy cascade, in which the enstrophy is transferred from large to small scales, and an inverse energy cascade, in which the energy is transferred from small to large scales [10]. Two power law behaviour are thus expected to describe this dual cascade, i.e.,

$$E_u(k) = C(\epsilon_\alpha)^{2/3}k^{-5/3}, \text{when } k_\alpha \ll k \ll k_f \text{ and } E_u(k) = C'(\eta_\nu)^{2/3}k^{-3}, \text{when } k_f \ll k \ll k_\nu \tag{1}$$

in which $E_u(k)$ is Fourier power spectrum of the velocity, ϵ_α is the energy dissipation by the Ekman friction, η_ν is the enstrophy dissipation by viscosity, k_f is the forcing scale, in which the energy and enstrophy is injected into the system, and k_α is the characteristic friction scale, k_ν is the viscosity scale. Specifically for the vorticity statistics, it is found experimentally that the vorticity is nonintermittent in the forward enstrophy cascade [12]. However, Nam *et al.* argued that if the Ekman friction α is relevant, the forward enstrophy cascade is then intermittent [11]. Note that in these studies, the classical structure-function (SF) analysis is involved to extract the scaling exponent $\zeta(q)$. More recently, Huang *et al.* found that the SF might be strongly influenced by energetic structures, e.g., high intensity vorticity in 2D turbulence [8]. The measured SF scaling exponent $\zeta(q)$ is then biased [7].

METHODOLOGY AND EXPERIMENTAL DATA

More recently, Huang *et al.* proposed a Hilbert-based methodology, namely Hilbert-Huang transform (HHT), to extract scaling exponents from a given scaling time series [9, 8, 7, 6]. The HHT is a data-driven method to decompose a given signal into a sum of Intrinsic Mode Functions (IMFs) without *a priori* basis, i.e., $\omega = \sum_{i=1}^{N} C_i + r_N$, in which C_i is IMF and r_N is residual [5, 9, 7]. The classical Hilbert spectral analysis is then applied to each IMF mode to extract instantaneous wavenumber k [5, 9, 8, 7]. This allows us to construct a pair of the extracted IMF modes and their instantaneous wavenumber, i.e., $[C_i, k_i]$ with $i = 1, \cdots N$ [6]. We therefore define a k-conditioned moment for $k_i = k$, i.e.,

$$\mathcal{L}_q(k) = \sum_{i=1}^{N} \langle C_i^q | k \rangle_x \sim k^{-\zeta(q)} \tag{2}$$

in which k is instantaneous wavenumber, $\langle\,\rangle_x$ is ensemble average over space and the scaling exponent $\zeta(q)$ is comparable with the scaling exponents provided by the classical SF approach [7, 6]. The Hilbert methodology has been successfully applied to Eulerian velocity [9], passive scalar [8, 7] and Lagrangian tracer particle [6] to characterize the intermittent nature of these processes. More details about this method can be found in Refs. [8, 7, 6].
The 2D vorticity version of the Ekman-Navier-Stokes equation, i.e.,

$$\partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = \nu \nabla^2 \omega - \alpha \omega + f_\omega \tag{3}$$

is solved by using a pseudospectral method with fully dealiased on a doubly periodic square domain of side $L = 2\pi$ at resolution $N^2 = 8192^2$ grid points. Here ν is the fluid viscosity, α is the Ekman friction and f_ω is an external forcing source. The main parameters are $\nu = 2 \times 10^{-6}$, $\alpha = 0.025$ and $k_f = 100$ [1]. More details of this database can be found in Ref. [1].

RESULTS AND DISCUSSION

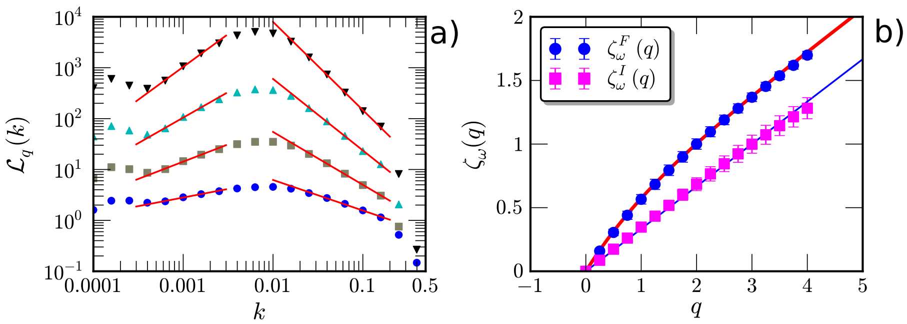

Figure 1. a) Measured Hilbert spectra for $q = 1$, 2, 3 and 4 (from bottom to top). Power law behaviors are observed on the ranges $4 \times 10^{-4} < k < 2 \times 10^{-3}$ and $2 \times 10^{-2} < k < 2 \times 10^{-1}$ respectively for inverse and forward enstrophy cascades. The solid line is the least square fitting. b) Measured scaling exponents for inverse (square) and forward (circle) cascades. The thick solid line is the log-Poisson like formula $\zeta^F_\omega = q/3 + 2/5(1 - (2/5)^q)$. The thin solid line is the fitting $\zeta^I_\omega(q) = q/3$.

Figure 1 a) shows the measured Hilbert spectra $\mathcal{L}_q(k)$ for $q = 1$, 2, 3 and 4. Graphically, we observe two power law behavior on the ranges $4 \times 10^{-4} < k < 2 \times 10^{-3}$ and $2 \times 10^{-2} < k < 2 \times 10^{-1}$ respectively for inverse and forward enstrophy cascades. The corresponding scaling exponents $\zeta_\omega(q)$ are then estimated on these ranges by using a least square fitting algorithm. The measured scaling exponents are shown in Fig. 1 b), in which the errorbar is the 95% fitting confidence. Visually, the measured scaling $\zeta^I_\omega(q)$ for inverse cascade is linear with q with a slope $1/3$, indicating a nonintermittent inverse enstrophy cascade. While, the measured forward scaling exponents $\zeta^F_\omega(q)$ are nonlinear with q, confirming the intermittent nature of the forward enstrophy cascade [11]. According to the log-Poisson proposal [4], we propose here a log-Possion like formula, i.e., $\zeta^F_\omega(q) = \frac{1}{3}q + \frac{2}{5}(1 - (\frac{2}{5})^q)$. The parameters $1/3$ and $2/5$ is from a fitting in the least square error sense. The above formula is illustrated as a thick solid line in Fig. 1 b), showing a good agreement between the formula and the measured $\zeta^F_\omega(q)$.

Note that the measured scaling exponents $\zeta_\omega(q)$ might be a function of α [11, 1, 2]. Therefore more database with different α should be investigated by using this novel methodology to characterize their intermittent nature in the Hilbert frame in future studies.

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