SPECTAL DIMENSION OF FRACTAL CLUSTERS IN TURBULENT FLOWS

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Abstract The *spectal dimension* is a generalisation of the fractal dimension which was introduced in a recent paper co-authored with H. R. Kennard and M. A. Morgan, [1]. It distinguishes between different fractals with the same Hausdorff dimension, by characterising the extent to which points are clustered onto lines or surfaces. This talk will discuss numerical investigations of the spectal dimension for inertial particles in random or turbulent flows. I shall also discuss the implications of the spectal dimension and its Renyi inspired generalisations for light scattering.

I will use two-dimensional models to illustrate the concept of the spectal dimension: the generalisations to higher dimensions are obvious. Consider the following approach to characterise the local structure of a point set. Take a given element, and consider an ellipse centred on this point, with its semi-minor axis of length ϵ and its semi-major axis of length $\delta = \epsilon^\alpha$, where $0 < \alpha < 1$. Then choose the orientation of the cover so that it maximises the number of other points which are contained in this ellipse. Denote the number of points under this optimally-oriented cover by $\mathcal{N}$. Repeat this for ellipses centred on other randomly selected points in the set, and compute the average value $\langle \mathcal{N} \rangle$ of the number of points which can be covered. In most of the examples of point-set fractals which were investigated [1], the mean number of points in this ellipse is found to have a power-law dependence:

$$\langle \mathcal{N}(\epsilon, \alpha) \rangle \sim \epsilon^{\beta(\alpha)} \,, \quad \delta = \epsilon^\alpha \tag{1}$$

where the exponent β depends upon α. In the case where $\alpha = 1$, the ellipse is a circle, so that this case reduces to a definition of the correlation dimension: $D_2 = \beta(1)$. I will show numerical examples, including simulations of fractal clusters for inertial particles in models for turbulent flows [2, 3, 4], demonstrating that the *spectal function* $\beta(\alpha)$ can distinguish between different fractal sets which have the same value of the correlation dimension $D_2 = \beta(1)$.

In the case of point set fractals which represent a physical distribution of particles, the existence of a highly anisotropic local structure has important physical implications for the scattering of light. If the particles have a strong tendency to accumulate along lines in two dimensions, or on planes in higher dimensions, light may scatter specularly from these structures. This motivates the investigation of the anisotropic covering with ellipsoids in the definition of the spectal dimension. Consider weak scattering of light with wavelength ϵ which propagates as a beam of width δ. When the path length for light scattered from different particles is large compared to ϵ, the scattering of light from $\mathcal{N}$ particles is incoherent, so that the contribution to the scattered intensity is $I \sim \mathcal{N}$. If, however, an ellipsoid of major axis δ and minor axis ϵ can be aligned to cover $\mathcal{N}$ particles, then there will exist directions where the path length difference is less than one wavelength, so that this set of $\mathcal{N}$ particles scatters light coherently. In these directions where the condition for specular reflection is satisfied by the optimal covering ellipsoid, there is a greatly increased intensity $I \sim \mathcal{N}^2$.

Understanding light scattering from dust particles dispersed in a turbulent medium will play an increasingly important role in observational astrophysics, and the generalisations of the spectal dimension will play an important role.

References

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