THE EFFECTS OF PRESSURE HESSIAN ON FLUID DEFORMATION

Yi Li[1]
[1]School of Mathematics and Statistics, University of Sheffield, S3 7RH, UK

Abstract The geometrical statistics of fluid deformation are numerically using direct numerical simulations. The analysis shows that the pressure Hessian is the leading cause to destroy the alignment between the longest axis of the material element and the strongest stretching eigen-direction of the strain rate. It also facilitates the alignment between the longest axis of the element and the intermediate eigen-direction of the strain rate during initial evolution, but tends to oppose the alignment later.

In this talk we look into the alignment between a material line element and the eigenvectors of the strain rate tensor. The alignment problem has been addressed in [2, 1, 4, 5, 3], among others. However, the effects of pressure have not been looked into in detail. We focus exactly on the pressure Hessian.

We denote the eigenvalues of the strain rate tensor S_{ij} as $\lambda_1^s \geq \lambda_2^s \geq \lambda_3^s$, and corresponding eigenvectors $\mathbf{e}_1^s$, $\mathbf{e}_2^s$, and $\mathbf{e}_3^s$. The eigenframe is called the S-frame. Let $\mathbf{\Omega}^s$ be the angular velocity of the eigenframe, we have

$$\epsilon_{ijk}\Omega_k^s = \frac{1}{\lambda_i^s - \lambda_j^s}\left[-\frac{1}{4}\omega_i^s\omega_j^s - P_{ij}^s + V_{ij}^s\right], \tag{1}$$

where Ω_k^s is the kth component of $\mathbf{\Omega}^s$ in the S-frame, P_{ij}^s is the pressure Hessian, V_{ij}^s is the diffusion term, and ω_i^s is the ith component of the vorticity. On the other hand, the evolution of a line element can be described by the deformation gradient $B_{ij} = \partial x_i/\partial X_j$, and the Cauchy-Green tensor $\mathbf{C}$ defind by $C_{ij} = B_{ik}B_{jk}$. Let $\lambda_1^c \geq \lambda_2^c \geq \lambda_3^c$ denote the eigenvalues of $\mathbf{C}$, $\mathbf{e}_1^c$, $\mathbf{e}_2^c$ and $\mathbf{e}_3^c$ the eigenvectors, and $\mathbf{\Omega}^c$ the angular velocity of the eigenframe (the C-frame), we have

$$\epsilon_{ijk}\Omega_k^c = \frac{\lambda_i^c + \lambda_j^c}{\lambda_i^c - \lambda_j^c}S_{ij}^c + \epsilon_{ijk}\frac{\omega_k^c}{2} \tag{2}$$

S_{ij}^c and ω_k^c are the components of the strain rate tensor and vorticity in the C-frame, respectively.

We are interested in the direction cosines $\alpha_i = |\mathbf{e}_1^c \cdot \mathbf{e}_i^s|$ $(i = 1, 2, 3)$. The equation for α_i can be found as

$$\frac{d\alpha_i}{dt} = \text{sign}(\mathbf{e}_1^c \cdot \mathbf{e}_i^s)(\mathbf{\Omega}^s - \mathbf{\Omega}^c) \cdot (\mathbf{e}_i^s \times \mathbf{e}_1^c). \tag{3}$$

which shows that the evolution of α_i is determined by the difference between the angular velocities projected on a direction perpendicular to both eigendirections. There are five contributions to the RHS of Eq. 3. We use R_o^s, R_p^s, and R_v^s to denote the contributions from the RHS of Eq. 1, and R_o^c and R_s^c to denote the vorticity and strain rate contributions, respectively, from Eq. 2. R denotes the total contribution, i.e., $R = \text{sign}(\mathbf{e}_1^c \cdot \mathbf{e}_i^s)(\mathbf{\Omega}^s - \mathbf{\Omega}^c) \cdot (\mathbf{e}_i^s \times \mathbf{e}_1^c)$.

The alignment between eigenvectors can be studied via the PDF of α_i, denoted as $P(\alpha_i)$. The evolution of $P(\alpha_i)$ is controled by $\langle R|\alpha_i\rangle$, whose product with $P(\alpha_i)$ gives the probability flux across α_i. Thus, where the gradient of $\langle R|\alpha_i\rangle P(\alpha_i)$ is negative (positive), the probability accumulates (disperses) hence PDF increases (decreases) with time.

Figure 1. The PDFs of $\alpha_i = |\mathbf{e}_1^c \cdot \mathbf{e}_i^s|$ at (a) $t = 3.68\tau_\eta$ and (b) $t = 7.35\tau_\eta$. Solid line: $i = 1$ (the strongest stretching direction of S_{ij}); dashed line: $i = 2$ (intermediate direction); dash-dotted line: $i = 3$ (contracting direction).

We present in Fig. 1 the PDFs $P(\alpha_i)$ calculated from DNS data for reference. Essentially it shows that initially $\mathbf{e}_1^c - \mathbf{e}_1^s$ alignment dominates, but the $\mathbf{e}_1^c - \mathbf{e}_2^s$ alignment dominates at later time, while $\mathbf{e}_1^c$ and $\mathbf{e}_1^s$ remains strong. These observations confirm what has been found in previous investigations.

Fig. 2(a) shows the contributions from the rotation of the S-frame. It shows that both vorticity (solid line) and the pressure Hessian (dashed line) work to strengthen the $\mathbf{e}_1^c - \mathbf{e}_2^s$ alignment, since their curves have negative gradients for large α_2. The contributions from the rotation of the C-frame is given in Fig. 2(b). Here vorticity tends to increase the alignment as

Figure 2. The probability fluxes $\langle R^*|\alpha_2\rangle P(\alpha_2)$ for $\mathbf{e}_1^c - \mathbf{e}_2^s$ alignment at $t = 3.68\tau_\eta$. (a) Solid line: R_o^s; dashed: R_p^s; dash-dotted: R_v^s ; dotted: the sum. (b) Solid: R_o^c; dashed: R_s^c; dotted: the sum.

Figure 3. The probability fluxes $\langle R^*|\alpha_2\rangle$ for $\mathbf{e}_1^c - \mathbf{e}_2^s$ alignment at (a) $t = 3.68\tau_\eta$ (b) $t = 1.22\tau_\eta$. Solid line: $R^* = R$ (total); dashed: $R^* = R_p^s$; dash-dotted: $R^* = R - R_p^s$.

well (solid line). It is, however, dominated by the counteracting contribution from straining (dashed line). Both figures show that, whilst vorticity tends to enhance the alignment, straining does the opposite.

To assess the importance of the pressure Hessian term, we compare it with the sum of the other four contributions, referred to as local effects. Fig. 3(a) shows the comparison at $t = 3.68\tau_\eta$. We observe that the magnitudes for the two are comparable, thus the pressure Hessian (dashed line) indeed has significant effects. The curve for local effects (dash-dotted) has a steep negative slope near $\alpha_2 = 1$, implying that they strongly prefer the $\mathbf{e}_1^c - \mathbf{e}_2^s$ alignment. The curve for pressure Hessian has a positive slope at $\alpha_2 = 1$. Thus, it does not prefer the perfect alignment between $\mathbf{e}_1^c$ and $\mathbf{e}_2^s$.

For α_2 around 0, the pressure Hessian contribution has a steeper positive slope. This means that pressure Hessian is more effective at aligning $\mathbf{e}_1^c$ and $\mathbf{e}_2^s$ when they are nearly perpendicular. This interpretation is collaborated by Fig. 3(b), where the comparison is made at an $t = 1.22\tau_\eta$. At this stage $\mathbf{e}_1^c$ dominantly aligns with $\mathbf{e}_1^s$ and tends to be perpendicular to $\mathbf{e}_2^s$. The figure shows that the contribution from pressure Hessian is much stronger, consistent with the above interpretation.

We also consider the results for the alignment between $\mathbf{e}_1^c$ and $\mathbf{e}_1^s$ (figures not shown). We compare the pressure Hessian and the total of other local effects for $t = 1.22, 3.68$ and $7.35\tau_\eta$, respectively. The comparison shows that the pressure Hessian is the main cause for the reduction of the $\mathbf{e}_1^c - \mathbf{e}_1^s$ alignment (with steep positve slope near $\alpha_1 = 1$). The effect persists throughout the evolution. The local effects initially also help reduce the alignment, but then turn to to enhance it at later time. The two contributions almost balance each other in the end.

CONCLUSIONS

To summarize, we show that the pressure Hessian is responsible for the misalignment between the long axis of the Cauchy-Green tensor and the stretching eigendirection of the strain rate. It facilitates the alignment between the long axis of the Cauchy-Green tensor and the intermediate eigendirection of the strain rate during initial evolution but reverses caurse later. The result highlights the important and subtle effects of the pressure Hessian.

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