ON THE FOUR-FIFTHS LAW IN MAGNETOHYDRODYNAMIC TURBULENCE

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Abstract The four-fifths law for two-point third-order longitudinal moments in three-dimensional (3D) incompressible magnetohydrodynamic (MHD) turbulence is examined. The examination was based on a generalization of the Kármán-Howarth-Kolmogorov equation for isotropic MHD turbulence to anisotropic MHD turbulence. Using direct numerical simulation data of 3D forced incompressible MHD turbulence without a uniformly imposed magnetic field in a periodic box, we quantify the viscous, forcing, anisotropy and non-stationary terms in the generalized equation. It is found that the influence of the anisotropic terms on the four-fifths law is negligible at small scales, compared to that of the viscous term.

It is thought that there is a certain kind of statistical universality at sufficiently small scales in fully developed magnetohydrodynamic (MHD) turbulence as well as hydrodynamic (HD) turbulence away from boundaries. A striking feature of this universality is the existence of exact statistical laws for homogeneous incompressible turbulence at sufficiently high Reynolds numbers. Such exact laws are rare. In three-dimensional (3D) incompressible homogeneous isotropic HD turbulence, we have Kolmogorov's 4/5 law [1], which can be derived from the Kármán-Howarth-Kolmogorov equation (hereafter referred to as the KHK equation).This 4/5 law is exact in the inertial subrange at infinitely large Reynolds number. The 4/5 law is extended to 3D incompressible MHD turbulence, using a generalized KHK equation for the two-point second-order velocity moments under the assumption that the flow is homogeneous and isotropic[2, 3]:

$$\langle\{\delta u_L(\boldsymbol{x},\boldsymbol{r},t)\}^3\rangle - 6\langle b_L^2(\boldsymbol{x},t)\delta u_L(\boldsymbol{x},\boldsymbol{r},t)\rangle = -\frac{4}{5}\bar{\epsilon}_t r, \qquad (1)$$

where δu_L is the longitudinal velocity difference between the points at $\boldsymbol{x}+\boldsymbol{r}$ and $\boldsymbol{x}$, which is defined as $\delta u_L = \{u_i(\boldsymbol{x}+\boldsymbol{r},t) - u_i(\boldsymbol{x},t)\}r_i/r$, b_L is the longitudinal component of the magnetic field defined by $b_L = b_i r_i/r$, $r = |\boldsymbol{r}|$, u_i is the ith component of velocity field, b_i is the ith component of the appropriately normalized magnetic field so that its dimension is the same as that of the velocity, $\langle\cdots\rangle$ denotes the ensemble average, $\bar{\epsilon}_t$ is the total energy dissipation rate per unit mass, and t is time. The summation convention over $\{1,2,3\}$ is used for the repeated subscripts i and j. This law is exact in the inertial subrange for infinitely large kinetic and magnetic Reynolds numbers. However, any real turbulence, in which the kinetic and magnetic Reynolds numbers and the scale range are finite, is not statistically isotropic, homogeneous, or stationary in a strict sense, owing to the influences of viscosity, external forcing, large-scale anisotropy, and so on. This law is examined using direct numerical simulation (DNS) data of 3D forced incompressible MHD turbulence without mean magnetic field in a periodic box under the assumption of flow isotropy and stationarity [3].

In this paper, we examine the influences of anisotropy, as well as large-scale forcing, viscosity, and non-stationarity, on the 4/5 law, Eq. (1), for 3D incompressible MHD turbulence in the absence of a uniformly imposed magnetic field in a periodic box. Emphasis is placed on the influence of anisotropy. We extend the generalized KHK equation for isotropic MHD turbulence to anisotropic MHD turbulence. The generalization procedure follows that in Ref. [4], in which a generalized KHK equation for incompressible anisotropic HD turbulence in a periodic box is derived to examine Kolmogorov's 4/5 law. The average $\langle\cdots\rangle$ is understood as the volume average over the fundamental periodic domain. We derive the following evolutions of the two-point second-order velocity moments, which is here called the generalized KHK equation for incompressible anisotropic MHD turbulence:

$$\langle\langle(\delta u_L)^3\rangle\rangle_r - 6\langle\langle b_L^2\delta u_L\rangle\rangle_r = -\frac{4}{5}\bar{\epsilon}_u r + I_\nu(r) + I_f(r) + I_t(r) + I_a(r), \qquad (2)$$

where $\langle\xi(\boldsymbol{r})\rangle_r$ denotes the average of $\xi(\boldsymbol{r})$ over $\boldsymbol{r}$ on the spherical surface of radius r with a center at $\boldsymbol{r}=\boldsymbol{0}$, $\bar{\epsilon}_u = \nu\langle\partial_j u_i\partial_j u_i\rangle + \langle b_i b_j \partial_j u_i\rangle$, $I_\beta = 3\int_0^r \hat{r}^3 H_\beta(\hat{r})\mathrm{d}\hat{r}/r^4$, $(\beta = \nu, f, t, a)$, $H_\nu(r) = 2\nu\partial_r\{\langle\langle\delta u_i\delta u_i\rangle\rangle_r\}$, $H_f(r) = \iiint_{|\hat{\boldsymbol{r}}|\le r} \langle\delta f_i^u(\hat{\boldsymbol{r}})\delta u_i(\hat{\boldsymbol{r}})\rangle \mathrm{d}^3\hat{\boldsymbol{r}}/(2\pi r^2)$, $H_t(r) = -\iiint_{|\hat{\boldsymbol{r}}|\le r}\partial_t\langle\delta u_i(\hat{\boldsymbol{r}})\delta u_i(\hat{\boldsymbol{r}})\rangle\mathrm{d}^3\hat{\boldsymbol{r}}/(4\pi r^2)$, $H_a(r) = H_u(r) + H_b(r)$, $H_u(r) = \{\partial_r\left(r^4\langle\langle(\delta u_L)^3\rangle\rangle_r\right)\}/(3r^3) - \langle\langle\delta u_i\delta u_i\delta u_L\rangle\rangle_r$, $H_b(r) = -2\{\partial_r\left(r^4\langle\langle b_L^2\delta u_L\rangle\rangle_r\right)\}/r^3 + 4\langle\langle b_L b_i\delta u_i\rangle\rangle_r$, $\partial_r = \partial/\partial r$, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\delta f_i^u = f_i^u(\boldsymbol{x}+\boldsymbol{r}) - f_i^u(\boldsymbol{x})$, f_i^u is the kinetic forcing, and ν is the kinematic viscosity. Using the magnetic energy equation, we find the following relation between $\bar{\epsilon}_u$ and $\bar{\epsilon}_t$: $\bar{\epsilon}_u = \bar{\epsilon}_t - \langle b_i f_i^b\rangle + \mathrm{d}\bar{E}_b/\mathrm{d}t$, where f_i^b is the magnetic forcing, and $\bar{E}_b = \langle b_i b_i\rangle/2$. If flow is strictly isotropic, $H_u = H_b = 0$, and then $I_a = H_a = 0$. Therefore, the term I_a expresses the degree of anisotropy. If all I_β $(\beta = \nu, f, t, a)$ are negligible in the inertial subrange, $\mathrm{d}\bar{E}_b/\mathrm{d}t = 0$, and if $f_i^b = 0$, then we obtain the 4/5 law which is consistent with Eq. (1) derived in Refs. [2, 3]. These works use a generalized KHK equation for homogeneous isotropic MHD turbulence and the evolution equation of mean kinetic and magnetic energies for the case that $f_i^b = 0$, in arriving Eq. (1). This equation (1) is in accordance with Eq. (7) for 3D case of Ref. [5] for generalized KHK equations for homogeneous isotropic MHD turbulence based on the Elsässer variables $z_i^\pm$, i.e., the evolution equations of $\langle\delta z_i^\pm\delta z_i^\pm\rangle$. The accordance shows that the latter does not contain the third-order moments arising from the induction equations.

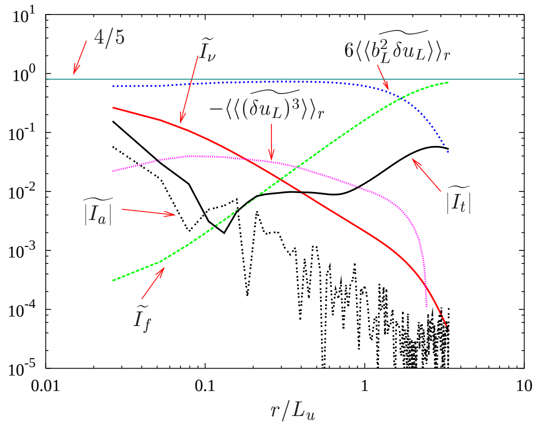

Figure 1. The r/L_u dependence of the normalized viscous term $\widetilde{I}_\nu$, forcing term $\widetilde{I}_f$, the magnitude of the normalized non-stationary term, $|\widetilde{I}_t|$, and the magnitude of the normalized anisotropic term, $|\widetilde{I}_a|$, together with the normalized third-order terms $-\langle\langle \widetilde{(\delta u_L)^3}\rangle\rangle_r$ and $6\langle\langle \widetilde{b_L^2 \delta u_L}\rangle\rangle_r$, where L_u is the integral length scale. The constant line denoting $4/5$ is plotted as a reference.

We quantify the influences of I_β ($\beta = \nu, f, t, a$) and of the third-order moments on the 4/5 law, using instantaneous DNS data of 3D forced incompressible MHD turbulence without a uniformly imposed magnetic field in a periodic box at the moderate kinetic and magnetic Taylor micro-scale Reynolds numbers, $R_\lambda^u = 158$ and $R_\lambda^b = 323$. The magnetic Prandtl number is set to one, and the number of grid points is 512^3. The forcing is imposed only on the large-scale velocity field. The DNS was performed until the field becomes quasi-stationary. The modulus of the time-derivative of $\bar{E}_b$ is very small, i.e., $|\mathrm{d}\bar{E}_b/\mathrm{d}t| \simeq 10^{-5} (\simeq 10^{-4}\bar{\epsilon}_t)$, and then $\bar{\epsilon}_u \simeq \bar{\epsilon}_t$.

Figure 1 shows that $6\langle\langle \widetilde{b_L^2 \delta u_L}\rangle\rangle_r$ is very dominant over $-\langle\langle \widetilde{(\delta u_L)^3}\rangle\rangle_r$ for all r, where $\widetilde{\cdot}$ denotes the normalization of $\cdot$ by $\bar{\epsilon}_u r$. This predominance is in accordance with the result obtained in Ref. [3], except at large scales. Taking into account the fact that $6\langle\langle b_L^2 \delta u_L\rangle\rangle_r$ and $-\langle\langle (\delta u_L)^3\rangle\rangle_r$ respectively result from the nonlinear terms $b_j \partial_j b_i$ and $u_j \partial_j u_i$, the predominance is also consistent with previous DNSs in Refs. [6, 7], where the flux for $b_j \partial_j b_i$ is dominant over that for $u_j \partial_j u_i$. It is seen that $6\langle\langle \widetilde{b_L^2 \delta u_L}\rangle\rangle_r$ is almost constant in the range $0.2 < r/L_u < 0.5$. This small departure is also consistent with the result in Ref. [3]. The normalized term $-\langle\langle \widetilde{(\delta u_L)^3}\rangle\rangle_r$ is comparable to the normalized forcing term $\widetilde{I}_f$ and the viscous term $\widetilde{I}_\nu$ in the range $0.17 \le r/L_u \le 0.38$. Except for this range, $-\langle\langle \widetilde{(\delta u_L)^3}\rangle\rangle_r$ is smaller than either the forcing term $\widetilde{I}_f$ or the viscous term $\widetilde{I}_\nu$. We observe that the normalized non-stationary term $|\widetilde{I}_t|$ is much smaller than $\widetilde{I}_\nu$ at small scales, and much smaller than $\widetilde{I}_f$ at large scales. However, it is also observed that around $r/L_u \simeq 0.3$, $|\widetilde{I}_t|$ is not very much smaller than $\widetilde{I}_\nu$ and $\widetilde{I}_f$, while $|\widetilde{I}_t|$ is one-order of magnitude larger than the normalized anisotropic term $|\widetilde{I}_a|$. We find that $|\widetilde{I}_a|$ is at least one-order of magnitude smaller than $\widetilde{I}_\nu$ except at large scales. This finding shows that the influence of these anisotropic terms on the 4/5 law is not significant compared to that of the viscosity, at least in our DNS. Therefore, the departure of the maximum value of $6\langle\langle \widetilde{b_L^2 \delta u_L}\rangle\rangle_r - \langle\langle \widetilde{(\delta u_L)^3}\rangle\rangle_r$ from 4/5 is mainly due to the influences of forcing, viscosity, and non-stationarity. We also examined the directional anisotropy defined by the departure of the third-order moments in a particular direction of $\boldsymbol{r}$ from the spherically averaged ones (figure omitted). Although the influence of the anisotropic term I_a on the 4/5 law is negligible, the influence of the directional anisotropy on the four-fifths law is suggested to be substantial, even in the case that we average the directional anisotropy over the three Cartesian directions, at least in the case studied here. This is in contrast to homogeneous quasi-isotropic HD turbulence examined in Ref. [4], where averaging $-\langle(\delta u_L)^3\rangle$ over the three directions is a good approximation of $-\langle\langle(\delta u_L)^3\rangle\rangle_r$. The reader interested in the detailed of this work may refer to Ref. [8].

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