Properties of the curvature tensor of streamtubes in turbulent flows

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Abstract Important features of the shape of streamtubes can be described by the first and second invariant of the curvature tensor $\partial t_i /\partial x_j$ where $\boldsymbol{t}$ is the unit vector tangent to the streamlines forming the tube. In three-dimensional turbulence, three different generic behaviors can be found, as well as four degenerate cases. For instance, the first invariant of $\boldsymbol{t}$ represents the relative change of the tube's cross-section. The joint PDF of the first and second invariant is evaluated from DNS, and the results stemming from the presented classification scheme are discussed. An attempt to reproduce the observed PDF from a restricted-Euler like model is described.

By definition, streamlines are tangent to the unit vector $\boldsymbol{t}$ of the velocity field. Streamtubes are constructed by tracing all streamlines starting at points forming a closed curve on an arbitrary surface nowhere tangent to a streamline. We define the curvature tensor $T_{ij} = \partial t_i /\partial x_j$ that in fact describes the shape evolution of the stream-tube cross-section for differentially thin streamtubes since it is the surface perpendicular to $\boldsymbol{t}$ (i.e. $\boldsymbol{t}$ is the cross-section's normal vector). In turn, the shape evolution of this surface describes important features of the shape of the streamtube. It can be shown that the first invariant of the curvature tensor (here denoted as H=-tr[T_{ij}]) is identical to the relative change of the tube's cross-section (as already derived by Schaefer[1] from the continuity equation), while the combination of the first and the second invariant (denoted as K=1/2(tr^2[T_{ij}]- tr[T_{ij}^2])) describes the geometry evolution of the tube. It can be shown [2] that the third invariant of the curvature tensor vanishes, as the third eigenvalue of the curvature tensor is identically zero. Thus, a phase diagram of in the H-K phase space provides useful information depending on the various regions (see Fig. 1). Positive values of H (right side of fig. 1) indicate a shrinking cross-section, while negative values lead to an increase (left side of fig.1). In combination with K, three different behaviors can be identified: Uniform shrinking/expansion (nodes), shrinking in one principal direction while expanding in the other one (saddle), or a spiraling movement around the tube's axis (foci). Note that the thick lines represent degenerate cases. The parabola separates real and complex eigenvalues.

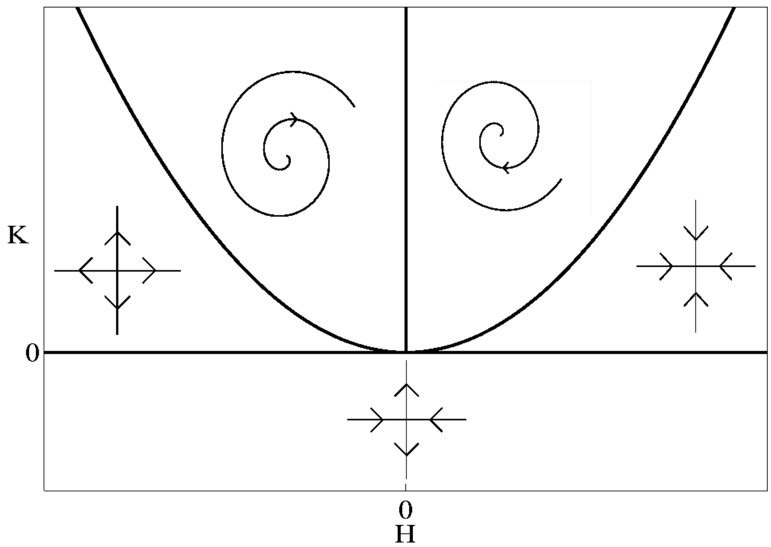

Figure 1. Phase plot of first (H) and second invariant (K) of the curvature tensor.

We measure K and H from Direct Numerical Simulation (DNS) of forced isotropic turbulence at moderate Reynolds numbers. Figure 2 shows the measured joint PDF of the two invariants. The very noticeable asymmetry (the long tail towards negative H) can be related to the skewness of the PDF of the velocity gradient and suggests that streamtubes expand rapidly while shrinking gently. It follows that expanding tube segments are shorter than segments where the cross-section decreases, which is in agreement with the findings of Wang [3]. The mean value of H is found to be zero, as required by continuity. The mean of H conditioned on K (the green curve in fig. 2) suggests that the expansion of streamtubes is mostly characterized by a widening of its cross-section in all directions (unstable node) or by spiraling

outwards (unstable focus). Shrinking streamtubes, on the other hand, are mostly determined by saddles, i.e. they are stretched in (at least) one direction and compressed in an other one.
From the exact dynamical equation for ***t*** derived from the Navier-Stokes equations, we show that by neglecting several pressure and viscosity-related terms (as in restricted Euler equations [4], we obtain a simplified dynamical equation that predicts trends consistent with the red parabola shown in Fig. 2. The solutions depend upon a parameter expressing a reference velocity, consistent with the fact that the streamtubes are not Galilean-invariant objects.

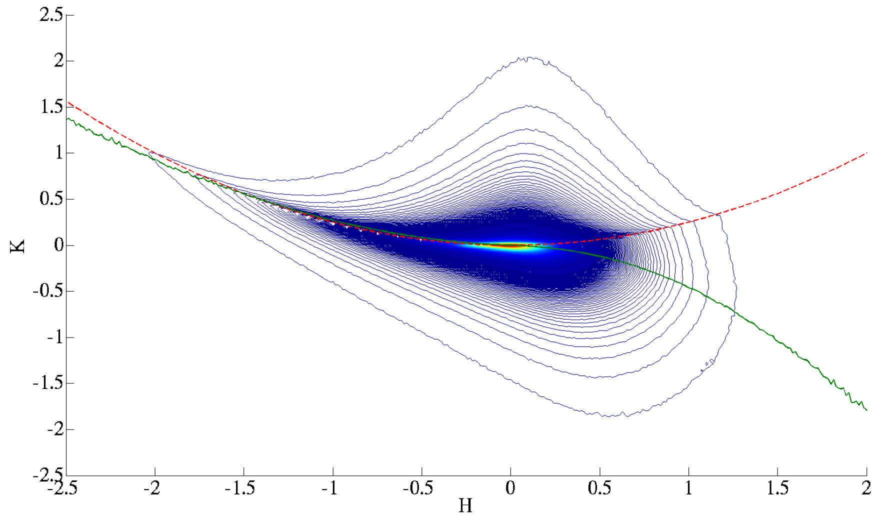

Figure 2. Measured JPDF of *H* and *K* obtained from DNS of forced isotropic turbulence. Red line is the parabola in fig. 1. Green line is the mean of *H* conditioned on *K*.

References

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