NUMERICAL INVESTIGATION OF THE ROLE OF DISSIPATION IN FLEXURAL WAVE TURBULENCE: FROM EXPERIMENTAL SPECTRA TO KOLMOGOROV-ZAKHAROV SCALINGS

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Abstract The Weak Turbulence Theory has been applied to flexural waves of thin elastic plates following the Foppl-von Kármán equations. The Kolmogorov-Zakharov spectrum $E_k^{KZ} \propto \phi^{1/3} k \log (k^*/k)^{1/3}$ predicted theoretically in the stationary case remains elusive in experiments. We performed some numerical simulations of the Foppl-von Kármán equations incorporating forcing and damping. When some truly transparent spectral window exists for intermediate wavenumbers, numerical simulations yields a spectrum close to the KZ-spectrum. This spectrum is steepened as the experimentally measured damping (not strongly localized in Fourier space) is incorporated and eventually resembles the experimental scaling $E_k^{EXP} \propto \phi^{1/2} k^{-0.2}$.

WAVE TURBULENCE THEORY AND EXPERIMENTAL RESULTS FOR ELASTIC PLATES

Phenomenology and hypotheses:

Weakly nonlinear waves systems (including but not limited to surface water waves, Alfven waves, sound waves, BEC description, *etc.*) might be considered as some descendants of their ancestor Hydrodynamical Turbulence, with whom they share a common energy cascade phenomenology: when the forcing and dissipative scales are clearly separated, an energy transfer might occur in the transparency window between them. This nonlinearities-induced energy flux usually builds up a power-law spectrum with respect to the wavenumber k and to the energy flux ϕ. The main difference between weakly nonlinear waves systems and hydrodynamical systems lies in the existence of a theoretical framework – the Wave Turbulence Theory (WTT) [1] – to derive statistical properties of the resulting chaotic wavefield, whereas similarity theory is usually used in Hydrodynamical Turbulence. One of the main achievement of WTT is the derivation of the so-called Kolmogorov-Zakharov (KZ) spectrum, associated to the stationnary out-of-equilibrium case. The derivation requires an infinite-size system, clearly separated forcing and dissipation and vanishingly small nonlinearities. From this last hypothesis, it is expected and required by the derivation that the solutions are slowly modulated waves and exhibit two characteristic timescales: a rapid one corresponding to the fast oscillations and a slow one for the energy transfer between waves. For real system, one might consider a third timescale due to dissipation that should be large compared to the two others. This hypothesis will be referred to as the "double timescale separation hypothesis" in the following.

Thin plates dynamical equations:

The deformation field z of thin elastic plates obeys the Föppl-Von Kármán equation in the limits of small slopes and small strains [2]:

$$\partial_{tt} z = -\frac{Eh^2}{12\rho(1-\sigma^2)} \Delta^2 z + \{z, \chi\} / \rho \tag{1}$$

$$\Delta^2 \chi = -\frac{E}{2\rho} \{z, z\} \tag{2}$$

In these equations, the properties of the material are described by its Young modulus E, its density ρ, its Poisson ratio σ; the thickness of the plate is denoted h; the curly brackets denote a differential bilinear operator $\{z, \chi\} = \partial_{xx} z \partial_{yy} \chi + \partial_{yy} z \partial_{xx} \chi - 2\partial_{xy} z \partial_{xy} \chi$. Bending accounts for the linear part of this equation and yields the dispersion relation $\omega = \sqrt{Eh^2/\left(12\rho(1-\sigma^2)\right)} k^2$ whereas the cubic term due to streching comes into play for higher amplitudes and is responsible for 4-waves interactions that yield an energy transfer between modes. Düring *et al.* applied WTT formalism to F-vK equations [3] and derived the corresponding KZ spectrum: $E_k^{KZ} \propto \phi^{1/3} k \log (k^*/k)^{1/3}$.

Experimental setup and results:

A thin stainless steel plate ($1\,\mathrm{m} \times 2\,\mathrm{m} \times 0.4\,\mathrm{mm}$) hangs under its own weight. Vibrations are excited by an electromagnetic shaker at a low frequency ($f_0 = 30$ Hz). Some shade-of-grey fringes are projected on the plate, recorded by an ultrafast camera and demodulated by an algorithm into a movie of the deformation of the plate. This space-time measure of the deformation field revealed much about the motion: although the motion is composed of waves [4] and although the double timescale separation hypothesis has been checked experimentally [6, 7], experimental spectra are in disagreement with KZ scalings: $E_k^{EXP} \propto \phi^{1/2} k^{-0.2}$ [4, 5].

NUMERICAL SIMULATION OF THE DYNAMICS OF A FORCED PLATE

The purpose of a numerical simulation of the dynamical equations is twofold. Firstly, the validity of some hypothesis needed in the derivation of the the Föppl-Von Kármán equations (e.g. the small strain limit) is not certain. Hence numerical simulations should confirm or infirm that these equations are a good starting point to reproduce the wave turbulence observed in real plates. Secondly, the physical parameters of the plates such as size or dissipation can be easily tuned in contrast with the real system.
The integration of the system (1-2) was performed with an order 2 Runge-Kutta algorithm in a pseudo-spectral method.

Simulation of realistic plates: reproducing experimental results
When the experimentally measured damping rate [6] $\gamma_k = a + bk^2$ is used, a good agreement is found between experimental and numerical deformation velocity spectra (see figure 1). In both cases the spectrum is significantly steeper than the KZ prediction with an exponential cutoff at small scales.

Localizing dissipation: recovering the KZ scaling
In order to meet the transparency window hypothesis, dissipation is decreased progressively toward zero over a spectral range $k < k^{cut}$ so that to localize the dissipation at $k > k^{cut}$. For a given injected power, figure 2 displays the evolution of the spectra as the dissipation is decreased. The spectra are seen to evolve from experimental-like spectra to KZ spectra.

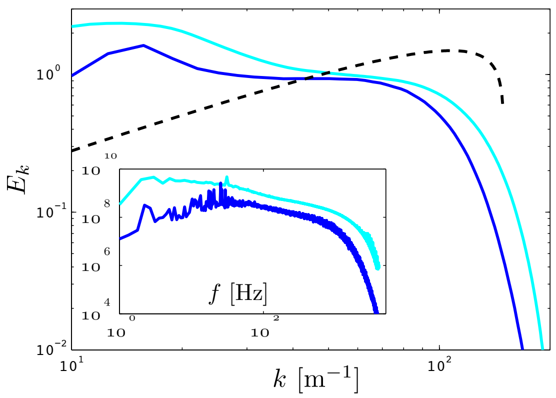

Figure 1. Velocity spectra E_k. Insert: frequency spectra. Cyan (light grey) lines: experimental spectra; blue (dark grey) lines: numerical spectra. Dashed line: KZ spectrum (eyeguide). Curves are vertically shifted for clarity.

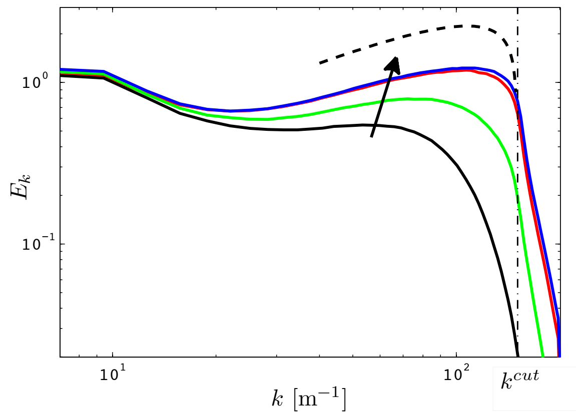

Figure 2. Plain color line: numerical spectra. Dissipation is decreased following the arrow direction; vertical dashed-dot line: k^{cut}. Dashed black line: KZ spectrum eyeguide.

CONCLUSION

By reproducing experimental results, this numerical simulation validates the description of our experimental system by the Föppl-von Kármán equation. The WTT treatment that yields the KZ spectra seems to fail due to the existence of some finite dissipation at any scale that prevents the energy flux to be constant over any range. This KZ scaling is nevertheless recovered when dissipation is removed, validating the WTT treatment for dissipationless F-vK equations.
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