SPECTRA IN GROSS-PITAEVSKII TURBULENCE WITHIN A SPECTRAL CLOSURE APPROXIMATION

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Abstract The spectra for Gross-Pitaevskii turbulence are obtained within a spectral closure approximation. The spectral closure approximation reduces to conventional weak wave turbulence (WWT) theory for weak non-linearity and it is also capable of deriving the spectrum for strong non-linearity. Spectra in turbulence with strong nonlinear are obtained for both the energy-transfer range and the particle-number-transfer range.

INTRODUCTION

Bose gases below critical temperatures are in the ordered phase where the order parameter $\psi := \langle\hat{\psi}\rangle$ is not 0. Here, $\hat{\psi}$ is the field operator and $\langle\cdot\rangle$ denote the vacuum expectation. The order parameter may depend on space and time as $\psi(\boldsymbol{x}, t)$ and its dynamics is described by the Gross-Pitaevskii (GP) equation[2, 5] (also called the nonlinear Schrödinger equation),

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = -\xi^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\psi - \mu\psi + g\psi^*\psi\psi, \tag{1}$$

under a certain approximation. Here, $\xi := \hbar/\sqrt{2m}$, $\hbar$ the Planck constant, m the mass of the particles, μ the chemical potential, g the coupling constant and the summation over repeated spatial coordinate indices are applied. Hereafter, we use the unit $\hbar = 1$.

The GP equation (1) can be interpreted as the equations of motion for a fluid by the use of Madelung transformation. We call the fluid quantum fluid. It is of interest to investigate whether the turbulence of quantum fluid is similar to the turbulence of ordinary fluid obeying Navier-Stokes equation in some sense. Superfluid component of liquid ^{4}He can be treated as a quantum fluid and some experiments[4] of liquid ^{4}He in Superfluid phase suggests that the energy spectrum is similar to the Kolmogorov energy spectrum $E(k) \propto k^{-5/3}$ in the ordinary fluid turbulence.

The statistics of quantum fluid turbulence has been investigated also by means of the numerical simulations of the GP equation [3, 7, 6]. The types of forcing and dissipation are different among the simulations, and the resulting spectra are also different among each other. Thus, it is not settled from the numerical simulations whether there exist the universal spectrum for turbulence obeying the GP equation.

Theoretically, the weak wave turbulence (WWT) theory[8, 1] is capable of analyzing the statistics of GP turbulence, when the non-linearity is small, i.e., the time scale $T_{\rm NL}$ characterizing the nonlinear dynamics is sufficiently longer than the time scale $T_{\rm L}$ of the linear wave dynamics. However, strong turbulence (ST) where $T_{\rm NL} \ll T_{\rm L}$ is out of the scope of the WWT theory.

In the present study, we attempt to derive theoretically the spectrum of the quantum fluid turbulence for the entire inertial range, not only the WWT region but the ST region, by means of a spectral closure approximation, or in other words, a two-point closure approximation.

SPECTRAL CLOSURE APPROXIMATION

Let $\psi_{\boldsymbol{k}}(t)$ be the Fourier transform of $\psi(\boldsymbol{x}, t)$ with respect to the coordinate variable $\boldsymbol{x}$. It is convenient to introduce a doublet

$$\begin{pmatrix}\psi^+_{\boldsymbol{k}}(t)\\ \psi^-_{\boldsymbol{k}}(t)\end{pmatrix} := \begin{pmatrix}\mathrm{e}^{\mathrm{i}(\xi^2k^2-\mu)t}\psi_{\boldsymbol{k}}(t)\\ \mathrm{e}^{-\mathrm{i}(\xi^2k^2-\mu)t}\psi^*_{-\boldsymbol{k}}(t)\end{pmatrix}. \tag{2}$$

By assuming statistical homogeneity in space, the two-point correlation function Q and the two-point response function G can be defined by

$$\langle\psi^\alpha_{\boldsymbol{k}}(t)\psi^\beta_{-\boldsymbol{k}'}(t')\rangle = Q^{\alpha\beta}_{\boldsymbol{k}}(t,t')(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{3}$$

$$\left\langle\frac{\delta\psi^\alpha_{\boldsymbol{k}}(t)}{\delta\psi^\beta_{\boldsymbol{k}'}(t')}\right\rangle = G^{\alpha\beta}_{\boldsymbol{k}}(t,t')(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{4}$$

where $\langle\cdot\rangle$ denotes an ensemble average and upper Greece indices denote $\{+,-\}$.

Closed equations for Q and G can be obtained by the method of renormalized expansion and truncation. The closure approximation is essentially GP equation equivalent of the direct interaction approximation (DIA) of the NS equation.

Here, in this abstract, we just denote the equations symbolically as

$$\frac{\partial}{\partial t}Q_{\boldsymbol{k}}(t,t') = \mathcal{A}_{\boldsymbol{k}}[Q,G](t,t'), \qquad \frac{\partial}{\partial t}G_{\boldsymbol{k}}(t,t') = \mathcal{B}_{\boldsymbol{k}}[Q,G](t,t'). \tag{5}$$

Let $\rho_{\boldsymbol{k}}(t)$ be the Fourier transform of the density field $\rho(\boldsymbol{x},t) := |\psi(\boldsymbol{x},t)|^2$. The two-point correlation function for $\rho_{\boldsymbol{k}}(t)$ is defined by

$$\langle \rho_{\boldsymbol{k}}(t)\rho_{-\boldsymbol{k}}(t')\rangle - \langle \rho_{\boldsymbol{k}}(t)\rangle\langle \rho_{-\boldsymbol{k}}(t')\rangle = Q^{\rho}_{\boldsymbol{k}}(t,t')(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{6}$$

and, within the closure approximation, its equation of motion can be written in terms of Q and G, i.e.

$$\frac{\partial}{\partial t}Q^{\rho}_{\boldsymbol{k}}(t,t') = \mathcal{C}_{\boldsymbol{k}}[Q,G](t,t'), \tag{7}$$

symbolically.

The particle-number per unit volume n and the kinetic and interaction energy per unit volume E_{K} and E_{I}, respectively, are given in terms of Q and Q^{ρ} as

$$n = \int_{\boldsymbol{k}} Q_{\boldsymbol{k}}(t,t), \qquad E_{\mathrm{K}} = \int_{\boldsymbol{k}} \xi^2 k^2 Q_{\boldsymbol{k}}(t,t), \qquad E_{\mathrm{I}} = \frac{g}{2}\left(\int_{\boldsymbol{k}} Q^{\rho}_{\boldsymbol{k}}(t,t) + n^2\right). \tag{8}$$

The particle number n and the energy $E := E_{\mathrm{K}} + E_{\mathrm{I}}$ are constants of motion.

SPECTRA IN THE CLOSURE

Let $T_{\mathrm{L}}(k) := \xi^{-2}k^{-2}$ be the time scale associated with the linear wave dynamics and $T_{\mathrm{NL}}(k)$ be the time scale associated with $Q_k(t,t')$ and $G_k(t,t')$. The wavenumber range where $T_{\mathrm{NL}}(k) \gg T_{\mathrm{L}}(k)$ [$T_{\mathrm{NL}}(k) \ll T_{\mathrm{L}}(k)$] is called weak wave turbulence (WWT) [strong turbulence (ST)] range. It is shown that the present closure equations reduce to the equations in the WWT theory[8, 1] for the WWT range.

We first consider the energy-transfer range, i.e. the energy flux $\Pi(K)$ from wavenumber range $k \le K$ to the wavenumber range $k > K$ is constant. In that case, we find from the closure equations that $T_{\mathrm{NL}}(k) = g^{-1}n^{-1}$. Then, we have ST range in $k \ll k_*$ and WWT range in $k \gg k_*$ with $k_* = \xi^{-1}g^{1/2}n^{1/2}$. Let the one-dimensional spectrum $F(k)$ be defined by

$$F(k) = \int_{\boldsymbol{k}'} \delta(k'-k)Q_{\boldsymbol{k}'}. \tag{9}$$

We find the scaling $F(k) \propto k^{-2}$ in the ST range and the scaling $F(k) \propto k^{-1}$, which is consistent with the WWT theory, in the WWT range.

Now we consider the particle-number-transfer range, i.e. the particle-number flux $\Pi_{\mathrm{n}}(K)$ from wavenumber range $k \le K$ to the wavenumber range $k > K$ is constant. We find $T_{\mathrm{NL,n}}(k) = g^{-1/2}|\Pi_{\mathrm{n}}|^{-1/2}$ for the time scale of $Q(t,t')$ and $G(t,t')$. We have $F(k) \propto k^{-1}$ in the ST range $k \ll k_{*,\mathrm{n}}$ and $F(k) \propto k^{-1/3}$, which is consistent with the WWT theory, in the WWT range $k \gg k_{*,\mathrm{n}}$ with $k_{*,\mathrm{n}} = \xi^{-1}g^{1/4}|\Pi_{\mathrm{n}}|^{1/4}$.

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