COMPARISON BETWEEN PRANDTL, NAVIER-STOKES AND EULER SOLUTIONS FOR A VORTEX DIPOLE IMPINGING ON A WALL

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Abstract We argue that d'Alembert's paradox (1749) is still unresolved for very large Reynolds number flows. Prandtl (1904) assumed that there exists a viscous boundary layer attached to the wall and predicted that the drag force dissipates energy there at a rate proportional to $Re^{-1/2}$. Kato (1984) proved that, in the limit of infinite Reynolds number, the energy dissipation rate tends to zero if and only if the solution of the Navier-Stokes equation converges towards the solution of the Euler equations (with the same initial data) and then occurs in a very thin boundary layer of thickness proportional to Re^{-1}. By performing direct numerical simulations of a dipole crashing into a wall we show that Kato's scaling is more appropriate than Prandtl's scaling as soon as the boundary layer detaches from the wall.

MOTIVATION

We propose to revisit the problem posed by Euler in 1748 for the Prize of Mathematics set by the Prussian Academy of Sciences concerning the resistance that fluid flows exert on solid bodies. In 1749 d'Alembert sent a contribution to this problem, where he proposed a partial differential equation (which was actually the precursor of Euler's equation). Unfortunately, d'Alembert was not able with it to explain the energy dissipation observed in fluid flows and he thus raised d'Alembert's paradox [1].

In the 19th Century Saint-Venant, and then Navier and Stokes, solved this paradox for the laminar flow regime, by showing the crucial role played by the fluid viscosity, which then lead to the Navier-Stokes equation published in 1822. In 1904 Prandtl [2] introduced the notion of boundary layer, assuming all viscous energy dissipation takes place only in the boundary layer, as long as it remains in contact with the body, and proposed a methodology to resolve d'Alembert's paradox for flows around streamlined bodies, such as airplane wings. To describe the viscous fluid flow in the boundary layer, whose thickness is inversely proportional to the square root of the Reynolds number, he derived the Prandtl's equation and succeeded to asymptotically match its solution with that of an inviscid fluid flow governed by Euler's equation outside the boundary layer.

RESULTS

In the work we address the following questions: does energy dissipate when the boundary layer detaches from the wall and how does this happen? A generic flow we consider the case of a vortex-dipole impinging onto a wall (see Fig.), that we study by Direct Numerical Simulation (DNS) at the highest possible resolution, up to 16384^2, to see how solutions behave in the vanishing viscosity limit (equivalent to Reynolds numbers Re tending to infinity).

Starting from the same initial flow and considering the same geometry, we compare the solutions obtained for Euler's equation, Prandtl's equation, and Navier-Stokes equation, using different numerical methods (a Fourier spectral scheme, combined with a volume penalization method to model the solid wall [3], and a high-order finite volumes scheme, with no-slip boundary conditions). In the vanishing viscosity limit, we observe the formation of a boundary layer of thickness scaling as $Re^{-1/2}$ (as predicted by Prandtl's 1904 theory), until a certain time t_D, where the boundary layer suddenly collapses down to a thickness at least as fine as Re^{-1}. The Prandtl equations cease to be valid for t tending to t_D, which manifests itself by the formation of a finite time singularity in their solution. For $t > t_D$, the boundary layer rolls up into a structure which detaches from the wall and, in accordance with a theorem of Kato [5], dissipates a finite amount of energy even at vanishing viscosity [4].

We isolate two regions where energy is actually dissipated: region A, a vertical slab inside the fluid domain, and region B, a square box around the center of the main structure that has detached from the wall at $t = 0.495$ (dotted box in Fig. , right). The energy dissipation rate is integrated respectively over the domain A or B and plotted versus Re in Fig. (left). It can be seen that in both cases the dependence on Re becomes weak for $Re \geq 2000$.

We show the evolution of the energy dissipation rate along three selected trajectories for $Re = 3940$ (Fig. , right). The first striking feature is that it displays a strong maximum for two particles which start from the wall (green and red curves), occurring when they are still in region A. In contrast, there is little dissipation along the third trajectory (blue curve), which starts away from the wall and never enters region A. At later times, energy dissipation goes back to much smaller values for one of the trajectories that approached the wall (red curve), while it is still one order of magnitude larger for the other one (green curve), because the particle is trapped inside the strong vortex produced at the wall (corresponding to region B at $t = 0.495$).

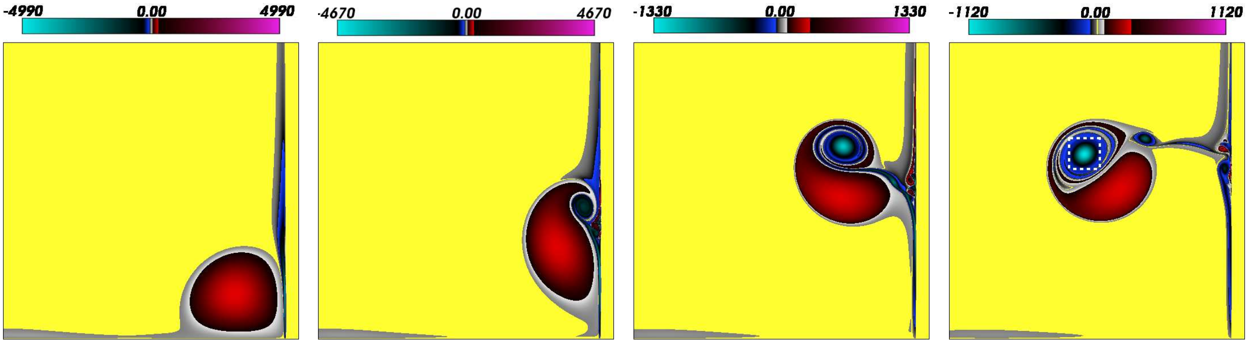

Figure 1. Navier-Stokes solution: vorticity at time $t = 0.36$, 0.4, 0.45 and 0.495 for $R_e = 7880$.

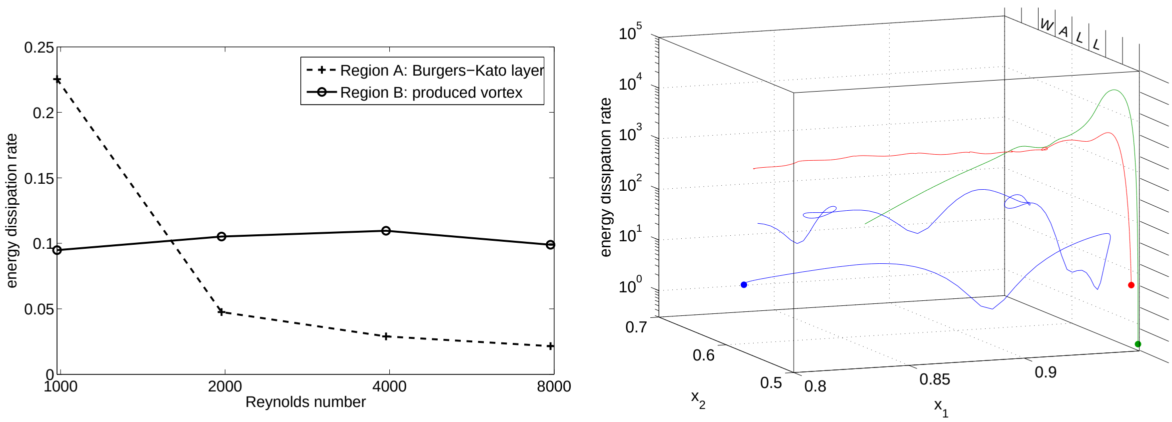

Figure 2. Left: Instantaneous energy dissipation as a function of Re at $t = 0.495$ in regions A and B (see text). Right: Energy dissipation rate versus particle position $(x_1(t), x_2(t))$ for $t \in [0.3, 0.495]$ along three Lagrangian trajectories, at $Re = 3940$. The circles indicate the positions at $t = 0.3$.

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