FUNCTIONAL RENORMALIZATION-GROUP APPROACH TO DECAYING TURBULENCE

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Abstract We reconsider the functional renormalization-group (FRG) approach to decaying Burgers turbulence, and extend it to decaying Navier-Stokes turbulence. The method is based on a renormalized small-time expansion, equivalent to a loop expansion, and naturally produces a dissipative anomaly and a cascade after a finite time. We explicitly calculate and analyze the one-loop FRG equations in the zero-viscosity limit as a function of the dimension. For Burgers they reproduce the FRG equation obtained in the context of random manifolds. Breakdown of energy conservation due to shocks and the appearance of a direct energy cascade corresponds to failure of dimensional reduction in the context of disordered systems. For Navier-Stokes in three dimensions, the velocity-velocity correlation function acquires a linear dependence on the distance, $\zeta_2 = 1$, in the inertial range, instead of Kolmogorov's $\zeta_2 = 2/3$; however the possibility remains for corrections at two- or higher-loop order. In two dimensions, we obtain a numerical solution which conserves energy and exhibits an inverse cascade, with explicit analytical results both for large and small distances, in agreement with the scaling proposed by Batchelor. In large dimensions, the one-loop FRG equation for Navier-Stokes converges to that of Burgers.

Describing Navier Stokes (NS) turbulence with the tools of statistical physics has remained a major challenge since Kolmogorov's dimensional arguments leading to the $E(k) \sim \bar{\epsilon}^{2/3} k^{-5/3}$ energy spectrum for the 3D energy cascade [8]. The simplest analytical method, Kraichnan's *direct interaction approximation* closure scheme [9] (equivalent to mode coupling) failed to recover Kolmogorov's prediction. There were numerous attempts to overcome these difficulties using a variety of methods, e.g. more refined closure schemes, large number of components, renormalization-group (RG), tetrad and shell models with various degrees of success. For the inverse cascade in 2D, due to an infinity of conserved quantities, and a simpler numerical modeling, more is known, e.g. the most recent analysis unveils a tempting connection to conformal field theory and SLE [3]. However, the latter remains based on numerics or speculative. The problem of N-dimensional Burgers turbulence, i.e. of a potential flow without pressure, exhibits similarities with NS turbulence, such as the existence of an inertial range which supports an energy cascade and the multi-scaling of the velocity moments. Although, as NS, it lacks a small control parameter and hence is non-trivial, it is simpler, since the Burgers equation can be integrated explicitly via the Cole-Hopf transformation [2]. A remarkable mapping to an elastic object in a quenched random potential maps the shocks in both a decaying or stirred Burgers velocity field to the jumps of the equilibrium position of the pinned elastic object (which is a point for decaying Burgers or a line for stirred Burgers) upon variation of an external field. This mapping was used to study the large-dimension N limit of *stirred* Burgers turbulence using replica symmetry breaking [4] and, more recently, of *decaying* Burgers turbulence [5]. The detailed statistics of shock cells which is obtained from these works is consistent with the physical expectation, and important open questions are now (i) whether this is a good starting point to perform an expansion towards finite N; (ii) whether it can inspire an approach to NS turbulence, a notably difficult problem.

Another powerful method able to handle singularities such as shocks and avalanches in disordered systems, which does not rely on large N, is the *Functional Renormalization Group* (FRG) [7]. The connection between the FRG and *decaying* Burgers turbulence was elucidated in [10, 11]. It turns out that the force felt by an elastic manifold of internal dimension d submitted to a random potential plus a quadratic well can be seen as a generalized velocity field: it satisfies an exact evolution equation which is a functional generalization of the decaying Burgers equation, where the role of time is played by the (inverse) curvature of the well. For $d = 0$ the manifold is a point and one recovers the standard Cole-Hopf representation of the Burgers equation. The hierarchy of equations relating n-point equal-time velocity correlation functions identifies with the (exact) hierarchy of FRG flow equations, and the loop expansion in the field theory corresponds to the (renormalized) small time expansion in the (generalized) Burgers problem. The amazing property is that this hierarchy *becomes controlled* in an expansion in $\epsilon = 4 - d$ around $d = 4$, which is the crucial property of the FRG approach to disordered systems. Hence Burgers turbulence, i.e. $d = 0$, becomes accessible via this expansion. Furthermore the physics of the generalized Burgers problem (i.e. of the manifold) has features which are independent of the parameter d. For instance, energy conservation for smooth flows is obtained as well as an infinite number of conserved quantities (the first property being called "dimension reduction" in the context of manifold, and the second corresponds to the non-renormalization of the moments of the so-called Larkin random force). Non-conservation of energy via shocks occurs for any d, and the dissipative anomaly at the heart of the energy cascade, i.e. the non-vanishing limit of the energy flux $-\partial_t \mathcal{E} = \bar{\epsilon} = \nu\langle(\nabla v)^2\rangle$ as $\nu \to 0$, is naturally captured by the FRG [11].

In this work [6] we investigate whether FRG-inspired methods can be developed to describe NS turbulence as well. Here, our scope is relatively modest and it should be seen as a first exploration of the FRG method into the domain of non-linear physics. We focus on the decaying Burgers and NS equations; however, the stirred case can also be studied within the same framework. We derive the one-loop FRG equations, first for Burgers in N-dimension and then for NS using small time expansion without mapping to any disordered system. At this stage, the method for NS is not a controlled perturbative expansion scheme, since there is no equivalent of the Cole-Hopf mapping. The method however does capture some of the physics of the singularities. We analyze the nature of the singularities at small distance. While our analysis is restricted to one loop, we also discuss possible extensions to higher loops. We study some features of the fixed-point solutions which correspond to a decaying turbulent state. The properties of the fixed point for the NS equation strongly depends on the dimension N. For $N \to \infty$, the FRG equations converge (at leading 1-loop order) to those of the decaying Burgers equation. Thus the 2-point velocity correlation function should grow linearly with distance, i.e. have a cusp. This cusp is also the only possible solution for the 3-dimensional FRG equation, at 1-loop order, in contradiction to experimental evidence. However, it is possible, that at second (2-loop) or higher order, new non-trivial fixed points emerge. In two dimensions the FRG equations allow for a non-trivial fixed point without cusp. The corresponding energy spectrum is shown in figure 1. This fixed point is compatible with the Batchelor-Kraichnan scenario [1] with an enstrophy anomaly $-\partial_t \frac{1}{2}\langle \omega_{\mathbf{u}t}^2 \rangle = \lim_{\nu\to 0} \nu \langle (\nabla\omega)^2 \rangle = \bar{\epsilon}_\omega$. The energy spectrum within the Batchelor-Kraichnan 2D enstrophy cascade is $E(k,t) \sim \bar{\epsilon}_\omega^{2/3} k^{-3}$, with $\bar{\epsilon}_\omega \sim 1/t^3$ in decaying case.

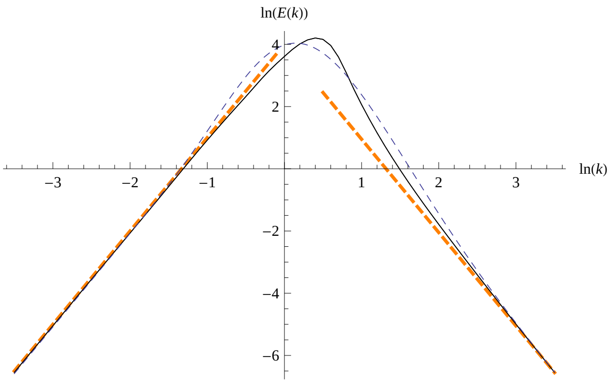

Figure 1. The energy spectrum in two dimensional turbulence computed using the fixed point of one-loop FRG equation.

References

[1] G.K. Batchelor. Computation of the energy spectrum in homogeneous two dimensional turbulence. *Phys. Fluids*, **12**:II–233, 1969.

[2] J. Bec and K. Khanin. Burgers turbulence. *Phys. Rep.*, **447**:1–66, 2007.

[3] D. Bernard, G. Boffetta, A. Celani, and G. Falkovich. Inverse turbulent cascades and conformally invariant curves. *Phys. Rev. Lett.*, **98**:024501, 2007.

[4] J.P. Bouchaud, M. Mezard, and G. Parisi. Scaling and intermittency in Burgers turbulence. *Phys. Rev. E*, **52**:3656–3674, 1995.

[5] P. Le Doussal, M. Müller, and K.J. Wiese. Avalanches in mean-field models and the Barkhausen noise in spin-glasses. *EPL*, **91**(5):57004, 2010.

[6] A.A. Fedorenko, P. Le Doussal, and K.J. Wiese. Functional renormalization-group approach to decaying turbulence. *arXiv:1212.2117*.

[7] D.S. Fisher. Interface fluctuations in disordered systems: $5-\epsilon$ expansion. *Phys. Rev. Lett.*, **56**:1964–97, 1986.

[8] A.N. Kolmogorov. On the energy distribution in the spectrum of a turbulent flow. *C.R. Acad. Sci. URSS*, **30**:301–305, 1941.

[9] R.H. Kraichnan. The structure of isotropic turbulence at very high reynolds numbers. *Journal of Fluid Mechanics*, **5**:497–543, 1959.

[10] P. Le Doussal. Finite temperature Functional RG, droplets and decaying Burgers turbulence. *Europhys. Lett.*, **76**:457–463, 2006.

[11] P. Le Doussal. Exact results and open questions in first principle functional RG. *Annals of Physics*, **325**:49–150, 2009.