EXPLICIT FORMULA OF ENERGY-CONSERVING, FOKKER-PLANCK-TYPE COLLISION TERM FOR SINGLE-SPECIES POINT VORTEX SYSTEM

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Abstract A kinetic equation for an unbounded two-dimensional point vortex system is considered. Similar to the Fokker-Planck equation, obtained collision term consists of a diffusion term and a drift (friction) term. It can be demonstrated that the collision term (a) conserves the system energy, (b) satisfies the H theorem that indicates the system temporally developing toward a thermal equilibrium state and (c) disappears locally in a locally equilibrium region with the same temperature. When system reaches a global equilibrium state, the collision term disappears all over the system.

POINT VORTEX SOLUTION AS A MICROSCOPIC SOLUTION

In the landmark paper published in 1949, Onsager proposed an application of statistical mechanics to two-dimensional (2D) point vortex system [8]. Here, he introduced a concept of “negative temperature". Since then, large research effort has been devoted to understand the negative temperature state both theoretically and numerically [1, 2, 4–7, 9–13].

In this paper, a kinetic Eq. for an unbounded 2D point vortex system is considered. Two-dimensional, inviscid Euler Eq. has a formal solution consisting of N point vortices:

$$\frac{\partial \hat{\omega}(\boldsymbol{r},t)}{\partial t} + \nabla\cdot(\hat{\boldsymbol{u}}(\boldsymbol{r},t)\hat{\omega}(\boldsymbol{r},t)) = 0, \tag{1}$$

$$\hat{\omega}(\boldsymbol{r},t) = \sum_{i}^{N} \Omega\delta(\boldsymbol{r}-\boldsymbol{r}_i) \tag{2}$$

where $\hat{\omega}(\boldsymbol{r},t)$ and $\hat{\boldsymbol{u}}(\boldsymbol{r},t)$ are vorticity and velocity fields in 2D plane $\boldsymbol{r} = (x, y)$, Ω is a positive constant and $\delta(\boldsymbol{r})$ is the Dirac delta function. For brevity, we shall omit the variables $\boldsymbol{r}$ and t. However, a macroscopic Eq. should have not a microscopic solution but a macroscopic (smooth) one. As the point vortex solution is regarded as a microscopic solution, it has a hierarchical inconsistency that the macroscopic Euler Eq. has a microscopic point vortex solution. So, we assume the Euler Eq. that has a microscopic point vortex solution, the counterpart of the Klimontovich Eq. in plasma physics.

An employed method to derive a collision term is Klimontovich formalism. It is assumed that the microscopic vorticity is decomposed in a macroscopic (ensemble-averaged) term and a fluctuation:

$$\hat{\omega} = \omega + \delta\omega, \qquad \omega \equiv \langle\hat{\omega}\rangle. \tag{3}$$

The velocity is also decomposed in the same manner. Inserting the decomposed variables into the microscopic Euler Eq. (1), we obtain a macroscopic Euler Eq. with a collision term and a linearized Eq. that will be needed in the sequel.

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega = -\nabla\cdot\langle\delta\boldsymbol{u}\delta\omega\rangle, \tag{4}$$

$$\frac{\partial \delta\omega}{\partial t} + \boldsymbol{u}\cdot\nabla\delta\omega = -\delta\boldsymbol{u}\cdot\nabla\omega \tag{5}$$

We assume that the collision term consist of a diffusion term and a drift (a.k.a. friction) term.

$$\langle\delta\boldsymbol{u}\delta\omega\rangle = -\mathsf{D}\cdot\nabla\omega + V\omega \tag{6}$$

Expressing D and V in the form of a perturbation expansion and gathering $O(\epsilon^0)$ terms for D and $O(\epsilon^1)$ terms for V, the analytical formula for the collision term $-\nabla\cdot\langle\delta\boldsymbol{u}\delta\omega\rangle$ is obtained, where ϵ is a small parameter of order $1/N$ or Ω with $N\Omega$ kept constant.

The obtained collision term is explicitly given by:

$$\begin{aligned}-\mathsf{D}\cdot\nabla\omega + \boldsymbol{V}\omega \;=\;& \Omega\int d\boldsymbol{r}'\int\frac{d\boldsymbol{k}}{(2\pi)^2}\int\frac{d\boldsymbol{k}'}{(2\pi)^2}\exp(i(\boldsymbol{k}+\boldsymbol{k}')\cdot(\boldsymbol{r}-\boldsymbol{r}')) \\ &\times\left[\pi\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')) - \frac{i\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2+d^2}\right]\frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}'}{|\boldsymbol{k}'|^2}\frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot(\omega'\nabla\omega - \omega\cdot\nabla'\omega')\end{aligned} \tag{7}$$

where

$$\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{(2\pi)^2}\int d\boldsymbol{k}\frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2}\exp(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')). \tag{8}$$

THREE IMPORTANT PROPERTIES OF THE OBTAINED COLLISION TERM

To reveal characteristics of the obtained collision term, we need to calculate the space average of the collision term. Space average is calculated over the small rectangular area Λ with sides both $2L$ located at $\boldsymbol{r}$.

$$\begin{aligned}\langle -\mathsf{D}\cdot\nabla\omega+\boldsymbol{V}\omega\rangle_S(\boldsymbol{r}) &\equiv \frac{1}{|\Lambda(\boldsymbol{r})|}\int_{\Lambda(\boldsymbol{r})} d\boldsymbol{r}''(-\mathsf{D}\cdot\nabla\omega+\boldsymbol{V}\omega)\\ &= \Omega\left(\frac{\pi}{L}\right)^2\int d\boldsymbol{r}'\int\frac{d\boldsymbol{k}}{(2\pi)^4}\pi\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}'))\frac{\hat{\boldsymbol{z}}\times\boldsymbol{k}}{|\boldsymbol{k}|^2}\frac{\hat{\boldsymbol{z}}\times\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot(\omega'\nabla\omega-\omega\nabla'\omega') \qquad (9)\end{aligned}$$

At first, let us examine if the collisional effect (9) disappears in local equilibrium state. Let us introduce a region inside which temperature is uniform and local equilibrium condition is satisfied [3]:

$$\omega_{\mathrm{leq}}=\omega_0\exp(-\beta\Omega\psi_{\mathrm{leq}}). \qquad (10)$$

Inserting Eq. (10) into the last term in Eq. (9), we find that the collision term equals zero, as the collision term has delta function $\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}'))$.

$$\frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot(\omega'\nabla\omega-\omega\nabla'\omega')=\frac{i\boldsymbol{k}}{|\boldsymbol{k}^2|}\cdot(\boldsymbol{u}'_{\mathrm{eq}}-\boldsymbol{u}_{\mathrm{eq}})\beta\Omega\omega_{\mathrm{eq}}\omega'_{\mathrm{eq}}=0 \qquad (11)$$

where $\boldsymbol{u}_{\mathrm{eq}}=-\hat{\boldsymbol{z}}\times\nabla\psi_{\mathrm{eq}}$ is used. Thus, the collisional effect disappears in the local equilibrium state.
Time derivative of the total system energy E is given by

$$\begin{aligned}\frac{dE}{dt} &= \frac{1}{2}\int d\boldsymbol{r}\int d\boldsymbol{r}'G(\boldsymbol{r}-\boldsymbol{r}')\left(\frac{\partial\omega'}{\partial t}\omega-\omega'\frac{\partial\omega}{\partial t}\right)\\ &= \Omega\frac{\pi^3}{L^2}\int d\boldsymbol{r}\int d\boldsymbol{r}'\int\frac{d\boldsymbol{k}}{(2\pi)^4}\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}'))\nabla\psi\cdot\frac{\hat{\boldsymbol{z}}\times\boldsymbol{k}}{|\boldsymbol{k}^2|}\frac{\hat{\boldsymbol{z}}\times\boldsymbol{k}}{|\boldsymbol{k}^2|}\cdot(\omega'\nabla\omega-\omega\nabla'\omega') \qquad (12)\end{aligned}$$

By permuting the dummy variables $\boldsymbol{r}$ and $\boldsymbol{r}'$ in Eq. (12) and taking the half-sum of the resulting expressions, we obtain

$$\begin{aligned}\frac{dE}{dt} &= \Omega\frac{\pi^3}{L^2}\int d\boldsymbol{r}\int d\boldsymbol{r}'\int\frac{d\boldsymbol{k}}{(2\pi)^4}\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}'))(\nabla\psi-\nabla'\psi')\cdot\frac{\hat{\boldsymbol{z}}\times\boldsymbol{k}}{|\boldsymbol{k}^2|}\frac{\hat{\boldsymbol{z}}\times\boldsymbol{k}}{|\boldsymbol{k}^2|}\cdot(\omega'\nabla\omega-\omega\nabla'\omega')\\ &= 0. \qquad (13)\end{aligned}$$

We conclude that the obtained collision term conserves the total system energy.
It is also shown that the obtained collision term satisfies the H theorem which indicates the system temporally develops toward the thermal equilibrium state. As there is no space left, we omit the details.

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