# On some applications of the self-avoiding randomly stretched vortices model for turbulence intermittencies

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*Abstract* In this work, it will be shown how a simple model of intermittency based on self-avoiding topological constraint on the evolution of a vortex line can be used to model the variations of the fluctuating local dissipation scale that has been recently introduced. Firstly, the evolution equation of the vorticity field is derived using a splitting of the velocity gradient evolution equation in its symmetric and antisymmetric parts. This allows a rigorous definition of the (random) effective stretching. Secondly, results obtained using the self-avoiding topological constraint on the vortex line are recalled. This leads to a log-stable distribution. Comparison is then made between latest state-of-the art DNS results. Then in a third part, a relationship is established between the fluctuating local dissipation scale and the local energy flux toward the small scale. This leads to a fully skewed to the right (log) stable distribution. Comparison is then made between recent results and the prediction of this model.

## Turbulent Effective Stretching

Let $A_{ij} = \frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij}$, be the velocity gradient tensor **A** decomposed into its symmetric **S** and its antisymmetric **Ω** parts (let us recall that both are orthogonal). The evolution equation of **A** then writes [1]:

$$\frac{dA_{ij}}{dt} = A_{ik}A_{kj} + \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \Delta A_{ij}$$

This leads to the following set of coupled equations for the evolution of the symmetric and antisymmetric part:

$$\frac{dS_{ij}}{dt} = S_{ik}S_{kj} + \Omega_{ik}\Omega_{kj} + \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \Delta S_{ij}$$

$$\frac{d\Omega_{ij}}{dt} = S_{ik}\Omega_{kj} + \Omega_{ik}S_{kj} + \nu \Delta \Omega_{ij}$$

Neglecting viscosity and using the fact that $\Omega_{ij} = \varepsilon_{ijk}\omega_k$ [2] where **ω** is the rotational of the velocity field, one gets, after some algebra, the evolution equation for the vorticity field:

$$\frac{d\boldsymbol{\omega}}{dt} = \mathbf{A}.\boldsymbol{\omega} - div(\mathbf{u})\boldsymbol{\omega} = \mathbf{B}.\boldsymbol{\omega}$$

where **B** is defined as the effective stretching tensor. Then defining **T** as the tangent unit vector to the vorticity field, then the vorticity intensity write $\omega = \mathbf{T}.\boldsymbol{\omega}$ and therefore:

$$\frac{d\omega}{dt} = T_i B_{ij}\omega_j = T_i u_{i,j}\omega_j - u_{i,i}\omega = b\omega$$

*b* is the local effective stretching which is both related to the local vorticity orientation and the local velocity gradient. In [3], the local effective stretching is considered as a time varying random variable $b_t$ i.e. a stochastic process. Then considering that vorticity lines are not self intersecting, it has been suggested a mapping between a vortex line and a self-avoiding walk (or SAW cf. figure 1).

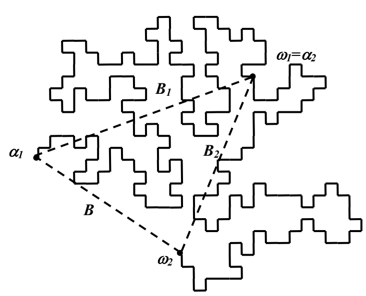


**Figure 1.** A sample 2D self-avoiding walk on a lattice.

It is then supposed that the scaling limit of the SAW is a Lévy stable process and that the scaling (or Flory) exponent $\nu$ of the gyration radius of the SAW could be related to the inverse of the stability index α of the Lévy law. Thought most of these results are conjectured in [3], some have been recently put into more solid ground by [4] or [5] (for some closely related case). All in all, it leads to a log-α-stable fully skewed to the left distribution for the distribution of the small scale vorticity. The value of the stability index can be assumed to be close to $\alpha$ = 1.70 and the scale parameter of the law follows from the relation:

$$\sigma_{\ln\varepsilon}^{\alpha} = \ln\left(\frac{\lambda}{\eta}\right)$$

where the distribution of turbulence dissipation is considered and $\lambda$ and $\eta$ are, respectively, the Taylor scale and the Kolmogorov scale.

**Application to the distribution of local dissipation scale**

The local dissipation scale is defined as [6], the scale $\eta$ where $\delta_\eta u \eta = \nu$ where $\delta_\eta u$ is the velocity increment over a distance $\eta$. It can be shown quite easily the following relation: $\partial_x u \approx (\delta_\eta u)^2 / \nu$ and $\varepsilon \approx (\delta_\eta u)^4 / \nu$. Therefore it can be inferred that $\varepsilon \approx \nu^3/\eta^4$ if $\varepsilon$ is log-stable so must be $\eta$. They have the same stability index but since $\log\eta = -\frac{1}{4}\log\varepsilon + ...$ their skewness parameter shall be opposite so that $\eta$ shall be fully skewed to the right. Moreover the scale parameter for the dissipation scale should be one fourth of the scale parameter of the dissipation rate. Using data from [7], this can be verified as can be seen in figure 2.

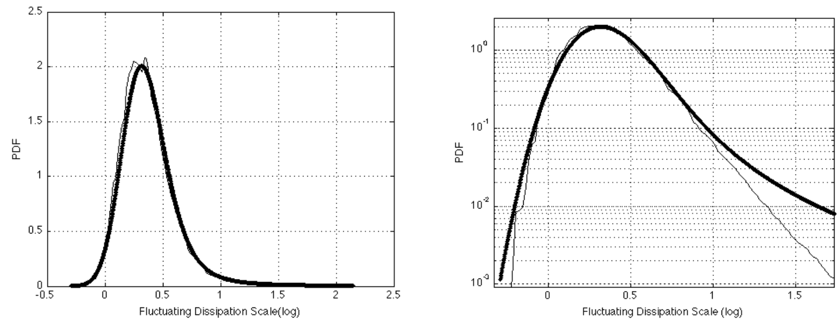


**Figure 2.** Comparison between the predicted log stable law (black dots) and the results of [7]. Note that the discrepancy increases for large value of $\eta$ in the log-log plot (on the right). Base of the logarithm is 10.

Values of the parameters of the stable laws are $\alpha$ = 1.70, $\beta$ = 1.00, $\sigma$ = 0.14 and $\delta$ = 0.40. As the Reynolds numbers in the pipe flow range from 24000 to 70000, the estimated values of $\sigma_{ln\varepsilon}$ range from 0.60 to 0.63, and therefore $\sigma_{ln\varepsilon}$ should range from 0.15 to 0.16, not far from the value used in figure 2. As the fluctuating dissipation scale has been non dimensionnalized by the Kolmogorov scale in [7], this induces a shift in the logarithmic scale. Therefore the value of the shift parameter is not studied.

**Conclusion**

The strength of the proposed modeling is twofold: firstly, it is rooted is simple physical principle (angular momentum conservation), secondly, it gives predictive results which can be compared to available results in the literature as can be seen in figure 2. It will be therefore also compared to recent DNS results [8] in a forthcoming work.

However there are also strong hypotheses underlying this modeling (such as the absence of vortex reconnection) which may fail on occasion. Search for these peculiar cases will also be the direction of future work as it will give interesting information about universality in turbulence.

**References**


[1] L. Chevillard and C. Meneveau Lagrangian dynamics and statistical geometric structure of turbulence *Phys. Rev. Lett.* **97**, 174501, 2006

[2] R. Aris Vectors, Tensors and the Basic Equations of Fluid Mechanics, Prentice-Hall, 1962

[3] N. Rimbert, Simple model for turbulence intermittencies based on self-avoiding random vortex stretching *Phys. Rev. E* **81**, 056315, 2010.

[4] M. Heydenreich Long-range self-avoiding walk converge to $\alpha$-stable processes *Annales de l'institut Henri Poincaré, Probabilité et Statistiques* **47** (1), 20-42, 2011

[5] L. Chen and A. Sakai Asymptotic behavior of the gyration radius for long-range self-avoiding walk and long-range oriented percolation *Annals of probability* **39**, 2, 507-548, 2011

[6] V. Yakhot Probability densities in strong turbulence *Physica D*, **215**, 166-174, 2006.

[7] S.C.C. Bailey, M. Hultmark, J. Schumacher, V. Yakhot and A.J. Smits Measurement of Local Dissipation Scales in Turbulent Pipe Flow *Phys. Rev. Lett.* **103**, 014502, 2009

[8] P.K. Yeung, D.A. Donzis and K.R. Sreenivasan Dissipation, enstrophy and pressure statistics in turbulence simulations at high Reynolds numbers *J. Fluid Mech.*, **700**, 5-15, 2012