WAVELET METHODS TO ELIMINATE RESONANCES IN THE GALERKIN-TRUNCATED BURGERS AND EULER EQUATIONS

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Abstract We employ a wavelet representation in the inviscid Burgers equation to analyse and treat the resonance phenomenon appearing in the Galerkin-truncated numerical solution to this problem. We use the Coherent Vorticity Simulation method to avoid the formation of the resonances, obtaining a appropriate regularized solution. The previously developed method with a complex wavelet frame is applied and expanded to embrace the use of real orthogonal wavelet basis, which we show to exhibit proper results only under the condition of adding a safety zone in wavelet space. We also apply the complex wavelet based method to the 2D Euler equation problem, showing it is able to filter the resonances in this case either.

INTRODUCTION

As discussed in [1], the Galerkin truncation of partial differential equations is a non-local operator in physical space, and localized structures, such as shocks in solutions of the Burgers equation, may act as sources of truncation waves that perturb numerical solutions at the cut-off scale. Furthermore, these waves may resonantly interact with the solution, generating strong perturbations localized in space that eventually spread and corrupt the numerical solution. In this work we will look for a wavelet representation to avoid the formation of the numerical resonances in the Fourier truncated Galerkin code to solve the inviscid Burgers equation.

To get further insight into the formation mechanism of the resonance phenomenon we make a continuous wavelet analysis to Galerkin-truncated solutions of the inviscid Burgers equation. Afterwards, we apply the wavelet filtering method analogous to Coherent Vorticity Simulation (CVS) [2, 3] to demonstrate that it is well suited to regularize its solution. For this, one solves the equation in Fourier space using a pseudo-spectral approach, but after each time step the solution is put in a complex wavelet frame, filtered with an iterative procedure introduced in [5], and then reprojected onto the Fourier basis for computing the next time step. We go further and propose the use of real orthogonal wavelets instead of the redundant complex wavelets from [2, 3]. We show that regularized solutions can only be obtained by keeping the neighbors of the retained coefficients, *i.e.*, adding a safety zone in wavelet coefficient space.

DUAL-TREE COMPLEX WAVELETS - KINGSLETS

The starting point is the inviscid Burgers equation written in conservation form

$$\partial_t u + \partial_x (u^2/2) = 0, \tag{1}$$

with the same harmonic initial condition as in [1]

$$u_0(x) = \sin(2\pi x) + \sin(4\pi x + 0.9) + \sin(6\pi x). \tag{2}$$

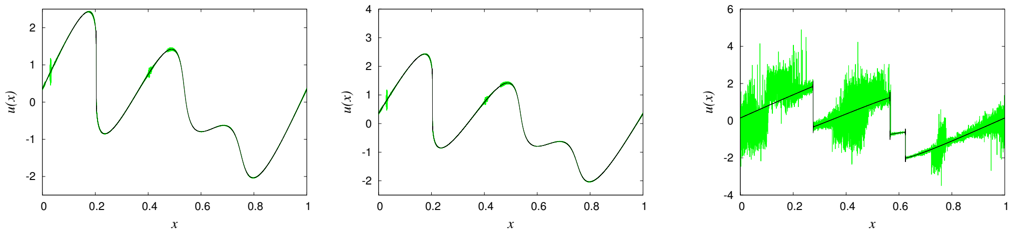
In [1] the authors observed that, when solving the Galerkin-truncated version of (1) with a pseudo-spectral code, one sees the appearance of small scale oscillations all over the solution right after the formation of the first singularity in the exact solution, followed by the emergence of two bulges around the points having the shock velocity due to a resonant interaction.

Our goal is to find a wavelet filtering method able to get rid of the resonances while keeping track of the dynamics of the untruncated Burgers solution. It turns out that the CVS method using the complex wavelet frame known as Kingslets [**?**], already applied to Burgers equation in [2, 3], is actually appropriate to the task. Fig. 1 compares the filtered solution with the Galerkin-truncated, similar to those obtained in [1], at three different time instants.

One sees that the whole dynamics of the Burgers equation is well preserved and has no trace of resonances, even for longer integration times when the Galerkin-truncated solution becomes perturbed.

REAL ORTHOGONAL WAVELETS

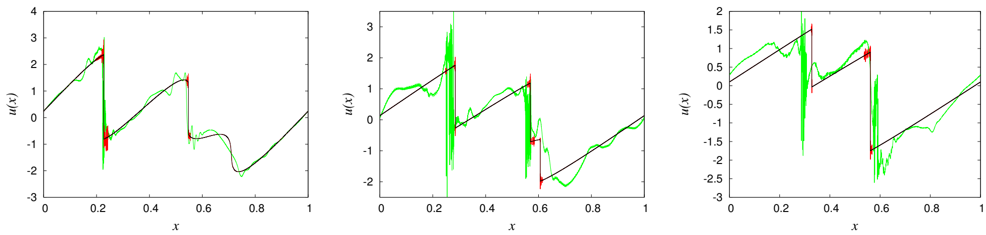
Although the Kingslet frame is well suited to get rid of resonances, it would be interesting to be able to use a non-redundant real orthogonal wavelet basis. Due to its lack of translation invariance, this kind of basis does not seem to work



(a) $t = 0.037$ (b) $t = 0.048$ (c) $t = 0.129$

Figure 1: (Color on-line) Green (gray): Galerkin-truncated solutions. Black: CVS filtered with Kingslets.

well with CVS and Burgers equation [3]. So following successful attempts in 2D and 3D CVS [6, 7, 8], we introduce the concept of a safety zone in wavelet space, that is, we also keep the wavelet coefficients neighbor in space and in scale to the ones kept by the original filter.
In Fig. 2 we show results using the Daubechies 12 wavelet basis. We see the significant improvement in the filtering capability of the code, comparing the cases with and without safety zone along with the analytical solution.



(a) $t = 0.067$ (b) $t = 0.143$ (c) $t = 0.200$

Figure 2: (Color on-line) CVS with periodic Daubechies 12 basis. Green (light gray): no safety zone. Red (dark gray): with safety zone. Black: analytical solution.

The naturally oscillating character of real wavelets and the lack of translation invariance still play a role generating small perturbations, but while the dynamics is lost when there is no safety zone, with shocks disappearing and huge oscillations developing, after the introduction of the safety zone it is very well preserved.
From this basics results, the quality of the approximations obtained for the different regularized solutions (also using other real wavelet basis) are assessed by computing a global error estimate. The compression level of the filtering methods presented are also compared.
As a starting point to a further analysis, we also apply the complex-valued wavelet method to the 2D incompressible Euler equations (as in [3]) to achieve the same kind of regularization there, since [1] also discusses the presence of resonances in this case. We show how the resonances are suppressed and how the profiles are strikingly similar, indicating a filter able to maintain the physical aspects of the solutions also in this case.

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