SOME STUDIES ON INCOMPRESSIBLE FLOWS ON GENERAL FIXED SMOOTH SURFACES

XiLin XIE, Qian SHI and Yu CHEN
Department of Mechanics & Engineering Science, Fudan University, Shanghai, CHINA

Abstract A novel kind of finite deformation theory with respect to continuous mediums whose geometrical configurations are general two dimensional surfaces have been developed by the authors. Generally, the theory is characterized by the interactions between the geometrical properties of the configurations/surfaces and deformations/motions. As a kind of applications, some studies on incompressible flows on general fixed smooth surfaces have been carried out. The new form of vorticity & stream-function algorithm is put forward through which wakes of a circular cylinder passing several kinds of curved surfaces have been studied numerically.

GENERAL THEORY OF CONTINUOUS MEDIUMS WHOSE GEOMETRICAL CONFIGURATIONS ARE TWO DIMENSIONAL SURFACES/RIEMANNIAN MANIFOLDS

Recently, amount of attentions have been paid to two dimensional flows including flows on soap films or spheres, finite vibrations of membranes, deformations of capsules and so on [1]. All these deformations/motions of continuous mediums can be considered as two dimensional flows since the geometrical configurations of continuous mediums can be taken as two dimensional smooth surfaces embedded in the three dimensional Euclidian space. The initial and current physical configurations with the descriptions of deformations in the parametric space are sketched in Fig.1. As compared to the general theory of continuum mediums, a novel kind of finite deformation theory of continuous mediums whose geometrical configurations are two dimensional surfaces have been developed by the authors [2]. It mainly includes definitions of initial and current configurations, deformation gradient tensor with its primary properties, deformation descriptions, transport theories and governing equations with respect to the nature conservation laws.

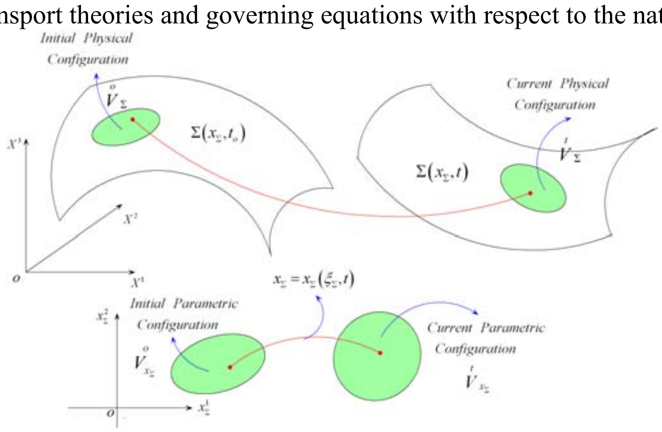

Figure 1. Sketch of the physical and parametric configurations for continuous mediums whose geometrical configurations can be considered as general surfaces/Riemannian manifolds. $x_\Sigma := x_\Sigma(\xi_\Sigma, t)$ is the description of deformation in the parametric space, x_Σ and ξ_Σ are Eulerian and Lagrangian coordinates respectively.

The mass conservation is represented as $\frac{d}{dt}\int_{\overset{t}{V}_\Sigma} \rho d\sigma = \int_{\overset{t}{V}_\Sigma} (\dot{\rho} + \rho\theta) d\sigma = 0$, where $\theta := \overset{\Sigma}{\nabla} \cdot V = \left(\frac{\partial}{\partial x_\Sigma^s} \overset{\Sigma}{g^s}\right) \cdot (V^\alpha g_\alpha)$. Subsequently, the differential form for mass conservation can be derived as follows

$$\dot{\rho} + \rho\theta = \frac{\partial \rho}{\partial t}(x_\Sigma, t) + \dot{x}_\Sigma^s \frac{\partial \rho}{\partial x_\Sigma^s}(x_\Sigma, t) + \rho\left(\overset{\Sigma}{\nabla}_s V^s - HV^3\right) = 0, \quad \dot{x}_\Sigma^s := \frac{\partial x_\Sigma^s}{\partial t}(\xi_\Sigma, t)$$

where $H := b_s^s$ denotes the mean curvature and $\overset{\Sigma}{\nabla}_s$ denotes the covariant derivative/Levi-Civita connention on Riemannian manifold. On momentum conservation, it is considered that the rate of the change of the momentum are due to the actions of surface tension, inner press, inner fraction and surface force, namely the following identity

$$\frac{d}{dt}\int_{\overset{t}{V}_\Sigma} \rho \vec{V} d\sigma = \int_{\overset{t}{V}_\Sigma} \rho \vec{a} d\sigma = \vec{F}_{ten} + \vec{F}_{pre}^{\text{int}} + \vec{F}_{vis} + \vec{F}_{sur}$$

is keeping valid, where

$$\vec{F}_{ten} := \oint_{\partial \overset{t}{V}_\Sigma} \gamma \vec{\tau} \times \vec{n} dl, \quad \vec{F}_{pre}^{\text{int}} := -\oint_{\partial \overset{t}{V}_\Sigma} p \vec{\tau} \times \vec{n} dl, \quad \vec{F}_{vis} := \oint_{\partial \overset{t}{V}_\Sigma} \mu (\vec{\tau} \times \vec{n}) \cdot \left[\left(\overset{\Sigma}{\nabla}_j V_i + \overset{\Sigma}{\nabla}_i V_j - 2b_{ij}V^3\right) \vec{g}^i \otimes \vec{g}^j\right] dl$$

are represented originally as curve integrals that can be transformed to surface integrals, $\vec{F}_{sur} = \int_{V_\Sigma^t} \vec{f}_{sur} d\sigma$ are represented as surface integrals, γ and μ denote the coefficients of surface tension and inner fraction/viscous respectively. It should be pointed out that the conservation of moment of momentum is naturally satisfied by the above mentioned actions. It can be derived that the component equations of momentum conservation for general compressible flows on general smooth surfaces take the following form

$$\begin{cases} \rho a_l = -\dfrac{\partial p}{\partial x_\Sigma^l} + \mu\left[g^{ij} V_i V_j + \overset{\Sigma}{\nabla}_l \left(\overset{\Sigma}{\nabla}{}^s V_s \right) + K_G V_l - 2\overset{\Sigma}{\nabla}_s \left(b_l^s V^3 \right) \right] + f_{sur,l} \\ \rho a_n = H(\gamma - p) + \mu\left[2b^{ij} V_i V_j - 2b_j^i b_i^j V^3 \right] + f_{sur,n} \end{cases}$$

, K_G denotes Gaussian curvature.

For flows on fixed surfaces, the acceleration takes the form $\vec{a} = \left[\dfrac{\partial V_l}{\partial t}(x_\Sigma, t) + V^s \overset{\Sigma}{\nabla}_s V_l \right] \vec{g}^l + \left[b^{ij} V_i V_j \right] \vec{n}$.

It is evident that the above governing equation of momentum conservation degenerates to the general compressible Navier-Stokes equation in the case that the geometrical configuration of continuous medium is a plat plane. Furthermore, we can extent the well-known vorticity & stream-function algorithm to the general incompressible flows on fixed smooth surfaces, that is

$$\frac{\partial \omega}{\partial t} + V^s \frac{\partial \omega}{\partial x_\Sigma^s}(x_\Sigma, t) = \frac{1}{Re}\left[g^{ij} \overset{\Sigma}{\nabla}_i \overset{\Sigma}{\nabla}_j \omega + \left(\varepsilon^{3kl} \frac{\partial K_G}{\partial x_\Sigma^k}(x_\Sigma, t) V_l + K_G \omega \right) \right] + \varepsilon^{3kl} \frac{\partial f_{sur,l}}{\partial x_\Sigma^k}(x_\Sigma, t)$$

$$\overset{\Sigma}{\Delta} \psi \triangleq g^{ij} \left[\frac{\partial^2 \psi}{\partial x_\Sigma^i \partial x_\Sigma^j}(x_\Sigma, t) - \Gamma_{ij}^k \frac{\partial \psi}{\partial x_\Sigma^k}(x_\Sigma, t) \right] = -\omega$$

The vorticity is defined as $\vec{\omega} \triangleq \left(\varepsilon^{3kl} \overset{\Sigma}{\nabla}_k V_l \right) \vec{n} =: \omega \vec{n}$ based on the deformation analysis, and the stream-function as usual is introduced through the relation $V^l \triangleq \varepsilon^{3ls} \partial \psi / \partial x_\Sigma^s (x_\Sigma, t)$ according to the continuity equation.

CASE STUDIES ON WAKES OF CYLINDERS ON SOME FIXED SMOOTH SURFACES

The wakes of a circular cylinder limited on some fixed smooth surfaces have been studied by the above mentioned algorithm. Some flow patterns are shown in Fig. 2 and Fig.3. Globally, the Von Karman vortex street still plays the dominate role but some local properties of vortices are modified by the curvatures of surfaces and surface fractions. The details will be revealed by the present study.

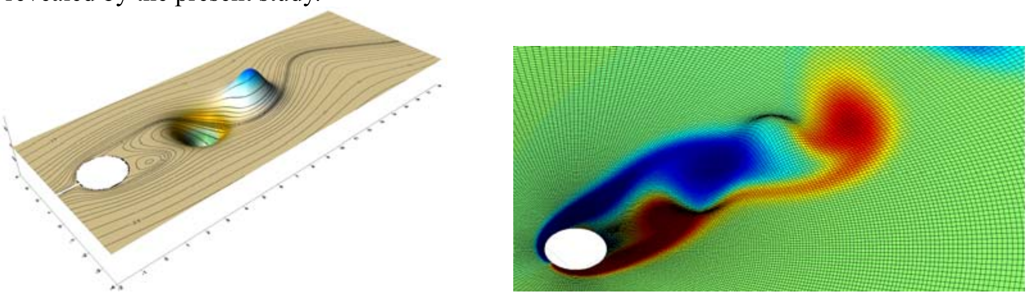

Figure 2. Snaps of flow patterns of a flow around a circular cylinder on a fixed surface: spatial distributions of stream-function and vorticity are shown by the left and right subplots respectively in the case of Re=100 and no surface friction is considered.

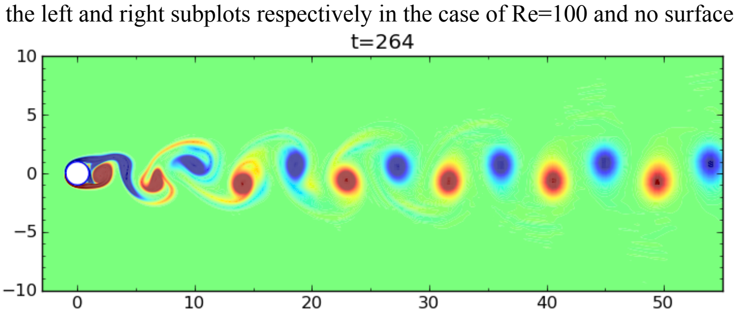

Figure 3. The planform of the spatial distribution of vorticity on the same surface as shown in Fig.2 in the case of Re = 500 and certain surface fraction is considered.

References

[1] G. Boffetta, R.E. Ecke. Two-dimensional turbulence. *Annu. Rev. Fluid Mech*. **44**: 427--451, 2012.

[2] X.L.Xie, Y. Chen, Q. Shi. Some studies on mechanics of continuous mediums viewed as differential manifolds. *Science China G* **56**: 1–25, 2013.