HALL MAGNETOHYDRODYNAMIC HARMONIC-HELICON ABSOLUTE EQUILIBRIUM

Jian-Zhou Zhu

WCI Center for Fusion Theory, NFRI, 169-148 Gwahak-ro, Yuseong-gu, Daejeon, Korea

Abstract L. Biferale et al.'s [Phys. Rev. Lett. **108** 104501 (2012)] discovery of inverse energy cascade in a pure helical wave system is consistent with Kraichnan's theory of helical turbulence [J. Fluid Mech. **59** 745 (1973)] which gives the absolute equilibrium energy spectral density $U(k) = 1/(\alpha+\beta k)\cancel{+1/(\alpha-\beta k)}$, with the contribution of helical waves with negative helicity being removed (shown by the slash in the formula). Such decomposition of Kraichnan's spectrum corresponds to the calculation of the harmonic-helicon mode absolute equilibrium in which further helical wave decomposition of the Fourier modes works to uncover the "degenerate states" and to present the finer spectral structure. Here, I present Hall magnetohydrodynamics harmonic-helicon absolute equilibrium which is easier and clearer for analysis and understanding of the statistical dynamics than previous studies, because, with the physical decomposition, the denominators reduce to second order polynomials instead of fourth order — see, e.g., S. Servidio, W. H. Matthaeus, and V. Carbone, Phys. Plasmas **15**, 042314 (2008).

INTRODUCTION

In a communication with C.-C. Lin in 1945, L. Onsager noted that the coefficients of the Fourier modes (harmonics) of hydrodynamic velocity field are "'momentoids' in the sense of Boltzman[n], and the theorem of equipartition would apply if their number were finite. Since this is not the case, we get a '[ultra]violet catastrophe' instead." [1] The theorem of equipartition was then rediscovered by T.-D. Lee [1] who introduced an upper limit of the wavenumbers and in particular showed the Liouville theorem explicitly for both hydrodynamics (HD) and magnetohydrodynamics (MHD). These giants, however, did not notice the importance of helicity which in general is also involved in the equipartition [2, 3]. Indeed, it was later when the importance of helicity was literally formulated (see, e.g., Moffatt and Tsinober [4] for a historical account), though helicity invariance is in the older Helmholtz-Kelvin theorem. Now, it has been widely realized that helicity plays an important role in dynamics in various situations, and tremendous progresses have been made, including notably those associated with helical wave decomposition [5, 2, 6, 7, 8, 9], among others.

HARMONIC-HELICON MODES

The Fourier helical mode decomposition for a 3D transverse vector field (velocity $\boldsymbol{u}$, transverse part of potential vector $\boldsymbol{A}$, magnetic field $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ and vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ etc.) in a cyclic box, with volume $\mathcal{V} = (2\pi)^3$, reads

$$\boldsymbol{v} = \sum_{\boldsymbol{k}} \hat{\boldsymbol{v}}(\boldsymbol{k}) e^{\hat{i}\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{c} \boldsymbol{v}^c = \sum_{\boldsymbol{k}}\sum_{c} \hat{v}^c(\boldsymbol{k})\hat{\boldsymbol{h}}_c(\boldsymbol{k}) e^{\hat{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{1}$$

Here $\hat{i}^2 = -1$ and the indexes "c" is for "chirality" ("+" or "-") with $c^2 = 1$; and the helical mode (complex vector) bases have the following properties $\hat{i}\boldsymbol{k}\times\hat{\boldsymbol{h}}_c(\boldsymbol{k}) = ck\hat{\boldsymbol{h}}_c(\boldsymbol{k})$, $\hat{\boldsymbol{h}}_c(-\boldsymbol{k}) = \hat{\boldsymbol{h}}_c^*(\boldsymbol{k}) = \hat{\boldsymbol{h}}_{-c}(\boldsymbol{k})$ and $\hat{\boldsymbol{h}}_{c_1}(\boldsymbol{k})\cdot\hat{\boldsymbol{h}}_{c_2}^*(\boldsymbol{k}) = \delta_{c_1,c_2}$ (Euclidean norm). We use Coulomb gauge ($\hat{\boldsymbol{A}}\cdot\boldsymbol{k} = 0$), so $\hat{i}\boldsymbol{k}\times[\hat{i}\boldsymbol{k}\times\hat{\boldsymbol{A}}(\boldsymbol{k})] = k^2\hat{\boldsymbol{A}}(\boldsymbol{k}) = \hat{i}\boldsymbol{k}\times\hat{\boldsymbol{B}}(\boldsymbol{k})$: The longitudinal component of $\boldsymbol{A}$ (with whatever gauge) is not involved here and later in the relevant calculations, so Coulomb gauge is not really necessary; we use this gauge just for convenience without loss of physics.

Waleffe [6] has looked into the detailed interacting triads of harmonic-helicon modes in Navier-Stokes for making theories. It is routinary to check that Liouville theorem and ruggedness of the quadratic invariants after Galerkin truncation, keeping only modes in $[k_{min}, k_{max}]$, which are true for all the models studied here. Noteworthily, Ditlevsen and Giuliani [7] studied the properties of separate +/- channels, which were critically reexamined by Chen et al. [8], emphasizing the exchanges between the two channels. It should be pointed out that further truncation of "+"- or "-"-modes, such as completely screening out one of the channels as done by Biferale et al. [9], for the inviscid system leads still to a conservative system. I will call each object being summed up in Eq. (1) harmonic helicon.

HARMONIC-HELICON ABSOLUTE EQUILIBRIUM

It can be shown [11] that the discovery of inverse energy cascade made by Biferale et al. [9] is consistent with Kraichnan's theory of helical turbulence [2] (K73) where the absolute equilibrium spectral density of energy is given by $U(k) = 1/(\alpha+\beta k)\cancel{+1/(\alpha-\beta k)}$. Here the contribution from the negative helicity, say, of left-handed helical modes is removed as designated by the slash, in accordance with Biferale et al.'s scheme of turning off one of the channels (they

[1] The letter was reproduced by G. Eyink and K. R. Sreenivasan [Review of Modern Physics, **78**, 87 (2006), p. 126], who did not discuss but later acknowledged this remarkable comment: Private communication (2008).

called it "decimation".) The left part is then exactly the same as the 2D electrostatic gyrokinetics [12] absolute equilibrium, on the diagonal of configuration-velocity scale space, given in [11] where interesting analogy between these two system was made. The point here is that the low helicity (gyrokinetic enstrophy in 2D gyrokinetics [11]) state corresponds to a negative temperature parameter α associated with energy and a positive temperature parameter β with helicity, like the case of 2D turbulence [10] (K67) but unlike K73: K73 considered the dynamics of mixed helical waves and that low helicity state corresponds to vanishing β with positive α and that equipartition of energy. A negative α condensates most of the energy in the modes with wavenumber close to the singular value $k_s = -\alpha/\beta$.

To be more specific, Galerkin truncated ideal hydrodynamic equations have the rugged invariants [2], kinetic energy $\mathcal{E}_K = \frac{1}{2\mathcal{V}} \int \boldsymbol{u}^2 d^3\boldsymbol{r} = \frac{1}{2}\sum_{\boldsymbol{k},c} |\hat{u}^c(\boldsymbol{k})|^2$ and kinetic helicity $\mathcal{H}_K = \frac{1}{2\mathcal{V}} \int \nabla \times \boldsymbol{u} \cdot \boldsymbol{u} d^3\boldsymbol{r} = \frac{1}{2}\sum_{\boldsymbol{k},c} ck|\hat{u}^c(\boldsymbol{k})|^2$. As K73, the Gibbs distribution[2] $\sim \exp\{-(\alpha\mathcal{E}_K + \beta\mathcal{H}_K)\}$ gives immediately the spectral densities of energy and helicity $U^c_K(k) = \frac{1}{\alpha + c\cdot\beta k}$, and $Q^c_K(k) = c \cdot kU^c_K(k) = \frac{c\cdot k}{\alpha+c\cdot\beta k}$. Summing up over c we get the K73 spectra.[3] In another word, the above results are simple decomposition of K73 spectra, but with clear physical meaning for either element. When the flow is dominated by helical modes with positive helicity or the positive helical modes are isolated as Biferale et al. [9] did, we may use only the $c = +$ component which is analogous to the K67 spectra.

I have performed similar calculations and analyses for various magnetohydrodynamic models [13], to demonstrate the harmonic-helicon power and to survey the properties of different helicities, including two-fluid MHD, Hall MHD, classical single-fluid MHD and electron MHD. Here, I present the Hall MHD results. This model is Hamiltonian with the canonical momenta (see, e.g., [14, 15]) $\boldsymbol{p}_i = m_e\boldsymbol{u} + q_i\boldsymbol{A}$ and $\boldsymbol{p}_e = q_e\boldsymbol{A}$, from which one can find the rugged invariants, magnetic helicity $\mathcal{H}_M = \frac{1}{2\mathcal{V}}\int \boldsymbol{A}\cdot\boldsymbol{B}d^3\boldsymbol{r}$ and "generalized" helicity $\mathcal{H}_G = \frac{1}{2\mathcal{V}}\int(\boldsymbol{u}\cdot\boldsymbol{B} + \frac{\epsilon}{2}\boldsymbol{\omega}\cdot\boldsymbol{v})d^3\boldsymbol{r}$, besides total energy $\mathcal{E} = \frac{1}{2\mathcal{V}}\int[\boldsymbol{u}^2 + \boldsymbol{B}^2]d^3\boldsymbol{r}$, and the spectral densities are:

$$U^c_K(k) = -4\,\frac{\alpha\,k + c\beta_M}{D^c_H},\ U^c_M(k) = -2\,\frac{(2\,\alpha + c\beta_G\,\epsilon\,k)\,k}{D^c_H},\ Q^c_M(k) = \frac{c}{k}U^c_M(k),\ \text{and}\ Q^c_G(k) = 2\,\frac{\beta_G\,k}{D^c_H} + c\frac{\epsilon}{2}kU^c_K(k),$$

with $D^c_H(k) = -4\,\alpha^2 k - c\cdot 4\,\alpha\,\beta_M - c\cdot 2\,\beta_G\,\epsilon\,k^2\alpha - 2\,\beta_G\,\epsilon\,k\beta_M + {\beta_G}^2 k$. The new notations follow the rule in the above and are explained by themselves. Summation over the c index produces Servidio et al. [16]: For comparison, I have used exactly the same form of invariants as theirs and that my α corresponds to their β, β_G to their γ and β_M to their θ.

DISCUSSION

With the above "pre-decomposed" Hall MHD harmonic-helicon absolute equilibrium, we can have much simpler and clearer analysis and understanding of the dynamics. ("Post-decomposition" of Servidio et al.'s [16] formulae is possible but formidable, not to mention the physics.) Otherwise, one has to deal with 4th order polynomials in the denominators. It is easier to derive the spectral properties, Hall effects, and so on and so forth. In particular, it can be used to guide direct numerical simulations with Biferale et al.'s [9] scheme or others, which we will do.

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[2] From now on, we will always use α for the energy related temperature parameter and β, with self-evident index when necessary, for the helicity, and we will always use the distribution of this style for the standard calculation; so, we will not repeatedly formulate and explain them.

[3] Here and below, we frequently use a $\cdot$ between c and the other quantities that are multiplied by it, just to highlight its effect.