STABILITY CRITERION OF STEADY ROTATION OF THE REGULAR VORTEX PENTAGON OUTSIDE A CIRCLE

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Abstract The nonlinear stability analysis of a stationary rotation of a system of five identical point vortices lying uniform on a circle of radius R_0 outside a circular domain of radius R is performed. The problem is reduced to the problem of equilibrium of Hamiltonian system with cyclic variable. The stability of stationary motion is interpreted as Routh stability. The conditions of stability, formal stability and instability are obtained subject to the parameter $q = R^2/R_0^2$.

INTRODUCTION

The stability problem of stationary rotation of the system of n identical point vortices lying uniformly on the circle (Thomsons vortex n-gons) was set by Kelvin (W. Thomson). There are generalizations of this problem to the cases of vortices inside and outside circular domain. All of these problems were solved by Havelock [1] in linear statement. It was found that corresponding linearized systems have exponentially growing solutions at $n \geq 8$ in the Kelvin's problem and at $n \geq 7$ in its generalizations. The exponential instability takes place at $2 \leq n \leq 6$ (the vortices inside and outside circle) in the case of certain values of the parameter. In the rest cases all of the eigenvalues of linearization matrix lie on a imaginary axis, so to solve of the problem is required nonlinear analysis.
The investigations of Kelvin problem at $n \leq 7$ was completed in the exact nonlinear statement in [2, 3, 4]. The results of nonlinear analysis in a circular domain were announced in [5], detailed for evenness of the number of vortices $n = 2, 4, 6$ in [6], and separately detailed for triangle [7] and pentagon [8].
The stability of Thomsons vortex n-gons ($n = 2, 4, 6$) outside a circular domain was studied in framework of unified approach in [9], and the stability of vortex triangle was studied in [11].
In this paper the nonlinear analysis of the stability of Thomsons vortex pentagon outside a circle is studied. The results is detailed in [10]. It based on results of A.D.Bruno, A.P.Markeev and A.G. Sokol'sky (see [12] and review [13]).

STATEMENT OF THE PROBLEM AND FORMULATION OF RESULTS

The motion of a system of n point vortices on a plane outside a circle of radius R is governed by the equations with the Hamiltonian (see, for example, [14])

$$H = -\frac{1}{4\pi} \sum_{1\leq j<k\leq n} \kappa_j\kappa_k \ln\left|z_j - z_k\right|^2 + \frac{1}{8\pi}\sum_{j=1}^{n}\sum_{k=1}^{n} \kappa_j\kappa_k \ln\left|R^2 - z_j\bar{z}_k\right|^2 - \frac{1}{4\pi}\sum_{j=1}^{n}\sum_{k=1}^{n} \kappa_j\kappa_k \ln\left|z_k\right|^2. \tag{1}$$

Here, $z_k = x_k + iy_k$, $k = 1, \ldots, n$ are complex variables; x_k, y_k are the Cartesian coordinates of the kth vortex; κ_k is the intensity of the kth vortex.
Here we assume that all vortices have the same intensity κ. The system with the Hamiltonian (1) has an exact solution

$$\begin{aligned} z_k &= e^{i\omega t}u_k, \quad u_k = R_0 e^{2\pi i(k-1)/n}, \quad k = 1, \ldots, n, \\ \omega &= \frac{\kappa}{4\pi R_0^2}\left(3n - 1 - \frac{2n}{1-q^n}\right), \quad q = \frac{R^2}{R_0^2} < 1. \end{aligned} \tag{2}$$

Therefore, a configuration of n identical vortices lying on the circle of radius R_0 at the vertices of a regular n-gon rotates with a constant angular velocity $\omega = \omega(q)$.
The stability of stationary rotation (2) of a vortex pentagon is analyzed. By a changes of variables the problem is reduced to the problem of a Hamiltonian system with a cyclic coordinate. Excluding the momentum corresponding to the cyclic coordinate, we obtain a reduced Hamiltonian system. The stability of the stationary rotation (2) is interpreted by us as Routh stability [15]. By Routh stability of the stationary rotation (2), we shall mean the Lyapunov stability of the equilibrium position of this reduced system. Correspondingly, instability of a vortex polygon will imply the Lyapunov instability of this equilibrium position.
Hereafter we use the concept of formal Routh stability, which is defined as a formal Lyapunov stability of a reduced system. A formal Lyapunov stability of equilibrium of a system means (see, for example, [12]) that there is a power series (possibly divergent) which is formally an integral of the system reaching a minimum at this equilibrium.
A criterion for the stability of the stationary rotation (2) of the vortex pentagon, is shown schematically in Fig. 1.

The resonances (see, [12, 13]) in which: q_{05} is double zero (diagonalizable case), q^* is resonance 1:2, q_{*5} is resonance 1:1 (nondiagonalizable case) correspond to the critical values of the parameter q

$$q_{05} = .3303989374, \quad q^* = .3333770174, \quad q_{*5} = .3345958365 \tag{3}$$

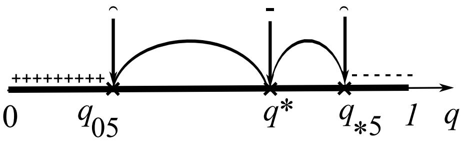

Figure 1. A stability criterion for Thomson's vortex pentagon outside circle: $q \in (0, q_{05})$ is Routh stability (++); $q \in [q_{05}, q^*) \cup (q^*, q_{*5}]$ – is formal Routh stability (solid arc); $q = q^*$ and $q \in (q_{*5}, 1)$ is instability (- -). The critical values of parameter q are given in (3).

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At $0 < q < q_{05}$ Routh stability follows from positive definiteness of Hamiltonian of linearized reduced system. At $q_{*5} < q < 1$ the instability was proved by Havelock [1]. The corresponding linearized system has exponentially growing solutions.

The proof of formal Lyapunov stability of reduced hamiltonian system of four degrees of freedom at $q \in (q_{05}, q_{*5}) \backslash q^*$ consisted in the verifying of the Bruno's theorem conditions [16]. At $q = q_{05}$ the critical case of double zero eigenvalues (diagonalizable case) take place in the stability problem. The formal stability follows from A.G. Sokol'sky results (see [13, 17])). In the critical case of double pairs of imaginary eigenvalues (jordan block) at $q = q_{*5}$ the proof of formal Routh stability repeats Sokol'sky arguments [18]. At $q = q^*$ the critical case of resonance $1 : 2$ takes place. Instability is proved by application of A.P. Markeev results [20, 13, 19].

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