THREE-VORTEX QUASI-GEOSTROPHIC DYNAMICS IN A TWO-LAYER FLUID

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Abstract The results presented here examine the quasi-geostrophic dynamics of a point vortex structure with one upper layer vortex and two identical bottom layer vortices in a two-layer fluid. In the present work, the existence conditions for stable stationary (translational and rotational) collinear two-layer configurations of three vortices are obtained. Small disturbances of stationary configurations lead to periodic oscillations of the vortices about their undisturbed shapes. These oscillations occur along elliptical orbits up to the second order of the Hamiltonian expansion. Analytical expressions for the parameters of the corresponding ellipses and for oscillation frequencies have been obtained.

We study fluid particle motion in the velocity field induced by a considered point vortex structure. The regular regimes are investigated, and the possibility of chaotic regimes (chaotic advection) under the effect of quite small nonstationary disturbances of stationary configurations has been shown. It is shown that regular and chaotic advection situations exhibit significant differences in both layers.

PROBLEM FORMULATION

The equations of motion of a point vortex system in a two-layer inviscid fluid rotating with an angular velocity $f/2$ (f is the constant Coriolis parameter) under the quasi-geostrophic approximation and assuming the rigid lid condition at the surface and flat bottom have the following non-dimensional form:

$$\dot{x}_j^\alpha = -\frac{h_j}{2\pi}\left\{\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{A_j}\kappa_j^\beta\frac{y_j^\alpha-y_j^\beta}{(r_{jj}^{\alpha\beta})^2}\left[1+r_{jj}^{\alpha\beta}\mathrm{K}_1\left(r_{jj}^{\alpha\beta}\right)\right]+\sum_{\beta=1}^{A_{3-j}}\kappa_{3-j}^\beta\frac{y_j^\alpha-y_{3-j}^\beta}{(r_{j(3-j)}^{\alpha\beta})^2}\left[1-r_{j(3-j)}^{\alpha\beta}\mathrm{K}_1\left(r_{j(3-j)}^{\alpha\beta}\right)\right]\right\}, \tag{1}$$

$$\dot{y}_j^\alpha = \frac{h_j}{2\pi}\left\{\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{A_j}\kappa_j^\beta\frac{x_j^\alpha-x_j^\beta}{(r_{jj}^{\alpha\beta})^2}\left[1+r_{jj}^{\alpha\beta}\mathrm{K}_1\left(r_{jj}^{\alpha\beta}\right)\right]+\sum_{\beta=1}^{A_{3-j}}\kappa_{3-j}^\beta\frac{x_j^\alpha-x_{3-j}^\beta}{(r_{j(3-j)}^{\alpha\beta})^2}\left[1-r_{j(3-j)}^{\alpha\beta}\mathrm{K}_1\left(r_{j(3-j)}^{\alpha\beta}\right)\right]\right\}. \tag{2}$$

Here $r_{ij}^{\alpha\beta} = \sqrt{\left(x_i^\alpha-x_j^\beta\right)^2+\left(y_i^\alpha-y_j^\beta\right)^2}$ is the distance between a vortex with dimensionless circulation κ_i^α and the number α within the ith layer and a vortex with dimensionless circulation κ_j^β and the number β within the jth layer ($\alpha,\ \beta = 1,\ 2,\ ...,\ A_j;\quad i,\ j = 1,\ 2$); $h_1 = h_2 = 1/2$ are the thicknesses of the top and bottom layers, respectively (the layers are numbered from top to bottom) and K_1 is a modifed Bessel function. We assume $A_1 = 1$, $A_2 = 2$ (i. e., we consider a special case of the three-vortex problem with one vortex in the top layer and two vortices in the bottom layer), $\kappa_2^1 = \kappa_2^2 = 1$, $\kappa_1^1 = \mu < 0$. The system (1)-(2), obtained for the first time by [Gryanik(1983)]. We suppose, that at the initial moment all three vortices lie on a straight line (without loss of generality, we can assume it to be the x axis), so that the top-layer vortex is situated in the origin of coordinates, as shown in figure 1. If the distances between vortex satisfy the equation

$$\frac{1}{2R}+\frac{2R(1+\mu)}{B(2R-B)}+\mathrm{K}_1(2R)+\frac{(2R+B\mu)\mathrm{K}_1(2R-B)-[2R(1+\mu)-B\mu]\mathrm{K}_1(B)}{2(R-B)}=0. \tag{3}$$

we have non-trivial non-symmetric collinear three vortex configurations [Sokolovskiy & Verron(2004)]. This collinear construction rotates as a solid body with the angular velocity

$$\omega = \frac{\mu+2}{4\pi(2R+B\mu)}\left[\frac{B+2R\mu}{2BR}-\mu\mathrm{K}_1(B)+\mathrm{K}_1(2R)\right].$$

REGULAR AND CHAOTIC ADVECTION AROUND THE PERTURBED STEADY STATES

We will focus now on the advection of fluid particles in a velocity field induced by the stationary vortex structures from the previous section. Figure 2a shows phase portraits for the stationary configuration in the bottom layer with $\mu = -2.0$. A set of heteroclinic separatrix, embracing the point vortices, forms in the indused velosity field.

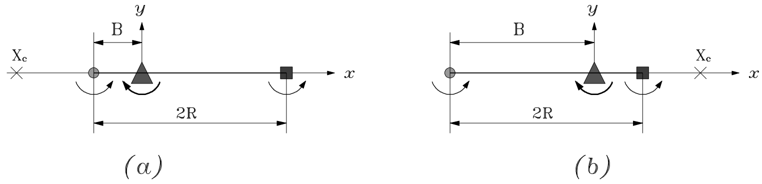

Figure 1. Scheme of the initial layout of vortices at $\mu = -2.5$: *(a)* $B < R$ and *(b)* $B > R$. Here and in the figures below, the triangle marks the position of the top-layer vortex and the circle and the square mark the positions of the bottom-layer vortices; X_c is the position of the vorticity center. The size of each marker is proportional to the absolute value of the intensity of the vortex. The arc arrows show the cyclonic or anticyclonic directions of the vortices.

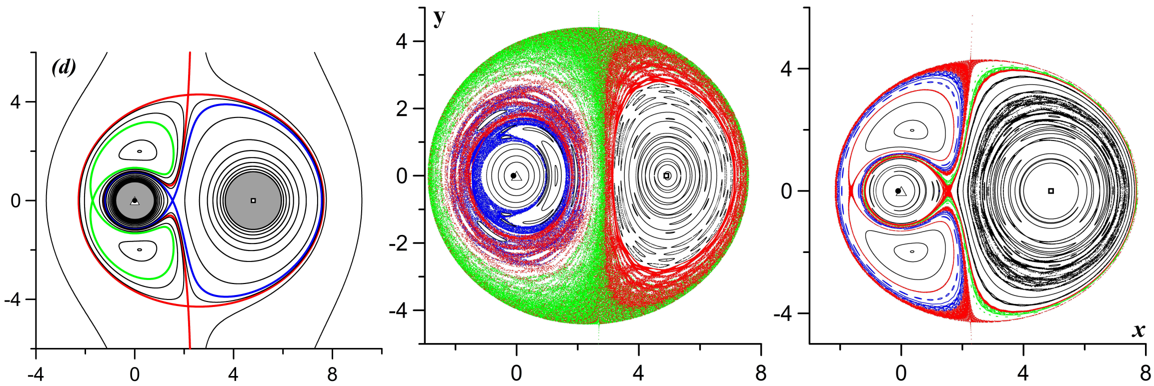

Figure 2. *(a)* Isolines of stream function in the bottom layer at $\mu = -2$ and the following another parameter values: $R = 2.4$, $B = 0.003131$; *(b)* Poincaré section for $\Delta R = 0.1$ for thetop layer; *(c)* Poincaré section for $\Delta R = 0.1$ for the bottom layer.

For simplification, we will consider perturbations preserving the zero component of the impulse of vortex system. We can consider the class of initial configurations with symmetric displacement of the peripheral vortices in the bottom layer, i. e., replace R by $R + \Delta R$ and B by $B + \Delta R$.

Figure 2 gives an example where the degree of chaotization is relatively small in the bottom layer and considerable in the top one. These differences were obtained by the choice of a sufficiently small perturbation amplitude, corresponding to high frequency perturbed oscillations.

However, the effect is largely governed by the dependence type of turnover frequency of fluid particles in the velocity field induced by a stationary configuration. In studies by [Izrailsky et al. 2008, Koshel et al. 2008], it is shown that when perturbations are not too small, the chaotization of phase portrait domains not adjacent to separatrices can be characterized by the overlapping degree of nonlinear resonances, which, in its turn, is largely determined by the width of the resonances' domains and the distance between neighbouring resonance domains. These parameters are determined by the derivative of the turnover frequency with respect to action, which is proportional to the derivative of the frequency with respect to coordinate.

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