CRITICAL WAVENUMBERS DIVIDING ENERGY SPECTRA BETWEEN WEAK AND STRONG TURBULENCE IN ELASTIC WAVES

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Abstract A weakly nonlinear spectrum and a strongly nonlinear spectrum coexist in a statistically-steady state in elastic wave turbulence. The critical wavenumbers which form the division between the weakly and strongly nonlinear spectra agree with the wavenumbers where the frequency increments due to the self-interactions are comparable with the linear frequencies.

INTRODUCTION

The wave turbulence statistics is investigated with direct numerical simulations (DNS) of the Föppl–von Kármán equation, which is the governing equation of the dynamics of the elastic waves propagating in a thin plate. The energy spectrum predicted by the weak turbulence theory $\mathcal{E}(k) \propto k$ appears in the large-wavenumber region. On the other hand, another spectrum $k^{-1/3}$ appears in the small-wavenumber region, where the nonlinearity is relatively strong. Both spectra coexist in one energy spectrum. (See Fig. 2 below.) In this study, the critical wavenumbers which form the division between the weakly and strongly nonlinear spectra are estimated by the comparison of the frequency given by the linear dispersion relation and that due to the self-interactions.

RESULTS

The FvK equation for the Fourier coefficient of the displacement $\zeta_{\boldsymbol{k}}$ and that of the momentum $p_{\boldsymbol{k}}$ under the periodic boundary condition is written as follows:

$$\frac{d}{dt}p_{\boldsymbol{k}} = -\frac{Eh^2}{12(1-\sigma^2)}k^4\zeta_{\boldsymbol{k}} - \frac{E}{2}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3} \frac{|\boldsymbol{k}\times\boldsymbol{k}_1|^2|\boldsymbol{k}_2\times\boldsymbol{k}_3|^2}{|\boldsymbol{k}_2+\boldsymbol{k}_3|^4}\zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}, \qquad \frac{d}{dt}\zeta_{\boldsymbol{k}} = \frac{p_{\boldsymbol{k}}}{\rho}, \tag{1}$$

where E, σ, ρ, and h are the Young's modulus, the Poisson ratio, the density, and the thickness of an elastic plate, respectively. The DNS of the elastic wave turbulence in the non-equilibrium statistically steady states where the small-wavenumber external forces and the large-wavenumber dissipation are added is performed by changing energy levels.
The FvK equation (1) is rewritten for the complex amplitude $a_{\boldsymbol{k}} = (\rho\omega_{\boldsymbol{k}}\zeta_{\boldsymbol{k}} + ip_{\boldsymbol{k}})/\sqrt{2\rho\omega_{\boldsymbol{k}}}$ as

$$\frac{d}{dt}a_{\boldsymbol{k}} = -i\omega_{\boldsymbol{k}}a_{\boldsymbol{k}} - \frac{iE}{8\rho^2}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3} \frac{|\boldsymbol{k}\times\boldsymbol{k}_1|^2|\boldsymbol{k}_2\times\boldsymbol{k}_3|^2}{|\boldsymbol{k}_2+\boldsymbol{k}_3|^4\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}_1}\omega_{\boldsymbol{k}_2}\omega_{\boldsymbol{k}_3}}}(a_{\boldsymbol{k}_1}+a^*_{\boldsymbol{k}_1})(a_{\boldsymbol{k}_2}+a^*_{\boldsymbol{k}_2})(a_{\boldsymbol{k}_3}+a^*_{\boldsymbol{k}_3}), \tag{2}$$

which means that $a_{\boldsymbol{k}}$ would rotate with the frequency $\omega_{\boldsymbol{k}} = \sqrt{\frac{Eh^2}{12(1-\sigma^2)\rho}}k^2$ in the negative direction in the phase space if the nonlinear terms were absent. The frequency spectrum $|\widetilde{a}_{\boldsymbol{k}}(\Omega)|^2$ is obtained from the time series of $a_{\boldsymbol{k}}(t)$ by the Fourier transforms. The numerically-obtained nonlinear frequency $\Omega_{\boldsymbol{k}}$ for each $\boldsymbol{k}$ is defined such that $|\widetilde{a}_{\boldsymbol{k}}(-\Omega_{\boldsymbol{k}})|^2$ is maximal.
Let us estimate the nonlinear frequencies to evaluate the nonlinearity. Among the nonlinear interactions, the combination of the wavenumbers including the self-interactions where one of $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ appearing in Eq. (1) is $\boldsymbol{k}$ is confined to $(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = (\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k},-\boldsymbol{k}')$ or $(\boldsymbol{k},\boldsymbol{k}',-\boldsymbol{k}',\boldsymbol{k})$. Then, the FvK equation (2) is rewritten as

$$\frac{d}{dt}a_{\boldsymbol{k}} = -i\omega_{\boldsymbol{k}}a_{\boldsymbol{k}} - i\omega^{\mathrm{s}}_{\boldsymbol{k}}(a_{\boldsymbol{k}} + a^*_{-\boldsymbol{k}}) + \text{non-self interactive terms}, \tag{3}$$

where

$$\omega^{\mathrm{s}}_{\boldsymbol{k}} = \frac{E}{4\rho^2\omega_{\boldsymbol{k}}}\sum_{\boldsymbol{k}'}\frac{|\boldsymbol{k}\times\boldsymbol{k}'|^4}{|\boldsymbol{k}-\boldsymbol{k}'|^4}\frac{|a_{\boldsymbol{k}'}|^2+|a_{-\boldsymbol{k}'}|^2+a_{\boldsymbol{k}'}a_{-\boldsymbol{k}'}+a^*_{\boldsymbol{k}'}a^*_{-\boldsymbol{k}'}}{\omega_{\boldsymbol{k}'}} = \frac{E}{2\rho\omega_{\boldsymbol{k}}}\sum_{\boldsymbol{k}'}\frac{|\boldsymbol{k}\times\boldsymbol{k}'|^4}{|\boldsymbol{k}-\boldsymbol{k}'|^4}|\zeta_{\boldsymbol{k}'}|^2 \tag{4}$$

is the frequency due to the self-interactions. Equation (3) is rewritten in the following simultaneous equation:

$$\frac{d}{dt}\begin{pmatrix} a_{\boldsymbol{k}} \\ a^*_{-\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} -i(\omega_{\boldsymbol{k}}+\omega^{\mathrm{s}}_{\boldsymbol{k}}) & -i\omega^{\mathrm{s}}_{\boldsymbol{k}} \\ i\omega^{\mathrm{s}}_{-\boldsymbol{k}} & i(\omega_{-\boldsymbol{k}}+\omega^{\mathrm{s}}_{-\boldsymbol{k}}) \end{pmatrix}\begin{pmatrix} a_{\boldsymbol{k}} \\ a^*_{-\boldsymbol{k}} \end{pmatrix} + \text{non-self interaction terms}. \tag{5}$$

Because the self-interactions preserve the inversion symmetry of the system, the frequency due to the self-interactions satisfies the relation $\omega^{\mathrm{s}}_{\boldsymbol{k}} = \omega^{\mathrm{s}}_{-\boldsymbol{k}}$. When $\omega^{\mathrm{s}}_{\boldsymbol{k}}$ is assumed to be constant in time, the eigenvalues of the matrix in the right-hand side of Eq. (5) determine the nonlinear frequency as $\omega^{\mathrm{NL}}_{\boldsymbol{k}} = \sqrt{\omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}}+2\omega^{\mathrm{s}}_{\boldsymbol{k}})}$.

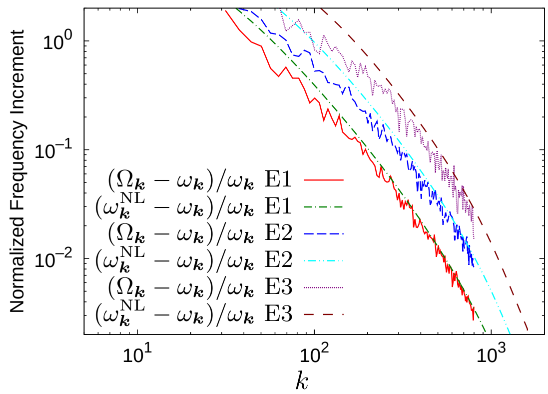

Figure 1. Nonlinear frequency increments normalized by the linear frequencies for three energy levels.

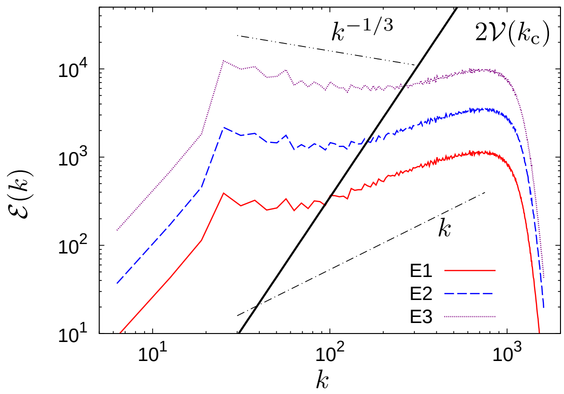

Figure 2. Energy spectra of three energy levels. The line shows $2\mathcal{V}(k_{\mathrm{c}})$.

Figure 1 shows the nonlinear frequency increments normalized by the linear frequencies. The frequency increments due to the self-interactions and those evaluated by the frequency spectra are respectively obtained as $\omega^{\mathrm{NL}}_{\boldsymbol{k}} - \omega_{\boldsymbol{k}}$. and $\Omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}}$. The frequency increments due to the self-interactions agree quite well with those evaluated by the frequency spectra in the low energy and in the large wavenumbers, where the nonlinearity is weak. On the other hand, in the high energy and in the small wavenumbers, the frequency increments due to the self-interactions overestimate the actual frequency increments. When the system is isotropic, the spectrum of the displacement $\langle|\zeta_{\boldsymbol{k}}|^2\rangle$ and the linear potential energy $\mathcal{V}(k)$ are related as $\langle|\zeta_{\boldsymbol{k}}|^2\rangle \approx 4\pi\mathcal{V}(k)/(\rho k \omega_{\boldsymbol{k}}^2)$. The weak turbulence theory is violated when the nonlinear frequency is comparable with the linear frequency. If the critical wavenumber k_{c} is defined as $\omega^{\mathrm{NL}}_{\boldsymbol{k}_{\mathrm{c}}} - \omega_{\boldsymbol{k}_{\mathrm{c}}} = \omega_{\boldsymbol{k}_{\mathrm{c}}}/2$, the potential energy at the critical wavenumber is given, under the assumption of the self-similarity $\mathcal{V}(k) \propto k$, as follows:

$$\mathcal{V}(k_{\mathrm{c}}) = \frac{5}{432}\frac{Eh^4}{(1-\sigma^2)^2}k_{\mathrm{c}}^3. \tag{6}$$

The factor $1/2$ results from the approximate upper limit below which the frequency increments due to the self-interactions agree with those evaluated by the frequency spectra in Fig. 1. Since $\mathcal{E}(k) \approx 2\mathcal{V}(k)$ in the weakly nonlinear wavenumbers, the critical wavenumber k_{c} satisfies $\mathcal{E}(k_{\mathrm{c}}) \approx 2\mathcal{V}(k_{\mathrm{c}})$. The energy spectra for the three energy levels are shown in Fig. 2. The line $2\mathcal{V}(k_{\mathrm{c}})$ intersects with the energy spectra at the critical wavenumbers which form the division between the weakly and strongly nonlinear spectra. It indicates that the critical wavenumbers estimated by the correspondence $\omega^{\mathrm{NL}}_{\boldsymbol{k}_{\mathrm{c}}} - \omega_{\boldsymbol{k}_{\mathrm{c}}} = \omega_{\boldsymbol{k}_{\mathrm{c}}}/2$ successfully mark the transition from the weakly nonlinear turbulence to the strongly nonlinear turbulence.