SCALE INVARIANT MODEL OF STATISTICAL MECHANICS AND QUANTUM MECHANICAL FOUNDATION OF TURBULENCE

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Abstract A modified statistical theory of turbulence based on a scale invariant model of statistical mechanics is described. Hierarchies of statistical fields from cosmic to *Planck* scale are examined. The predicted velocity profiles for turbulent boundary layer over a flat plate at four consecutive scales of LED, LCD, LMD, and LAD are shown to be in close agreement with the experimental observations. Invariant *Schrödinger* equation is derived from invariant *Bernoulli* equation. Energy spectrum of isotropic stationary turbulence is shown to follow invariant *Planck* energy distribution law.

SCALE INVARIANT MODEL OF STATISTICAL MECHANICS

It is well known that the methods of statistical mechanics can be applied to describe physical phenomena over a broad range of scales of space and time from the exceedingly large scale of cosmology to the minute scale of quantum optics [3] as schematically shown in Fig. 1.

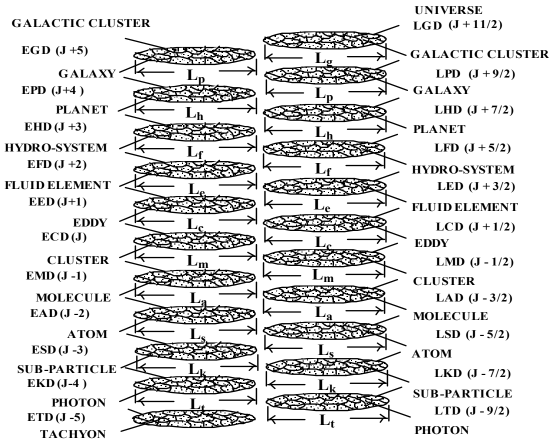


Figure 1. A scale invariant view of statistical mechanics from cosmic to tachyon scales.

For each statistical field, one defines particles that are point-mass or "*atom*" of the field, *elements* that are ensemble of "*atoms*", and finally system that is the ensemble of all "*elements*" (Fig.1). The atomic and system velocities of scale β $(\mathbf{u}_\beta, \mathbf{w}_\beta)$ are the most-probable speeds of the lower and upper adjacent scales [3].

INVARIANT FORMS OF CONSERVATION EQUATIONS

Following the classical methods of *Maxwell*, *Boltzmann*, and *Enskog*, the invariant definitions of density and atomic, element, and system velocities are introduced to arrive at the scale-invariant forms of mass, thermal energy, linear and angular momentum conservation equations at scale β [3]

$$\frac{\partial \rho_\beta}{\partial t} + \nabla.(\rho_\beta \mathbf{v}_\beta) = \Omega_\beta \qquad (1)$$

$$\frac{\partial \varepsilon_\beta}{\partial t} + \nabla.(\varepsilon_\beta \mathbf{v}_\beta) = 0 \qquad (2)$$

$$\frac{\partial \mathbf{p}_\beta}{\partial t} + \nabla.(\mathbf{p}_\beta \mathbf{v}_\beta) = -\nabla.\mathbf{P}_{ij\beta} \qquad (3)$$

$$\frac{\partial \boldsymbol{\pi}_\beta}{\partial t} + \nabla.(\boldsymbol{\pi}_\beta \mathbf{v}_\beta) = 0 \qquad (4)$$

involving the volumetric density of mass ρ_β, thermal energy $\varepsilon_\beta = \rho_\beta \tilde{h}_\beta$, linear momentum $\mathbf{p}_\beta = \rho_\beta \mathbf{v}_\beta$, and angular momentum $\boldsymbol{\pi}_\beta = \rho_\beta \boldsymbol{\omega}_\beta$. Also, Ω_β is the chemical reaction rate and $\tilde{h}_\beta = \int_0^T c_{p\beta} dT_\beta$ is the absolute enthalpy.

QUANTUM MECHANICAL FOUNDATION OF TURBULENCE

It is shown that the energy spectrum of stationary isotropic turbulence is governed by the invariant *Planck* energy distribution law [3]

$$\frac{\varepsilon_j dN_j}{V} = \frac{8\pi h}{u_j^3} \frac{\nu_j^3}{e^{h\nu_j/kT} - 1} d\nu_j \tag{5}$$

Kolmogorov-Obukhov $\kappa^{-5/3}$ law is found to be a local feature, valid only within inertial subrange, of the more universal *Planck* energy spectrum at large wave numbers. With cluster and eddy energies $\varepsilon = mu^2$ and $E = \sum \varepsilon = \alpha\varepsilon$, and cluster velocity given as $u = \nu\lambda = 2\pi\nu/\kappa$ such that $du = d(2\pi\nu/\kappa) \propto \kappa^{-2} d\kappa$ where $\kappa = 2\pi/\lambda$ is the wave number, one arrives at $dE = F(\kappa)d\kappa = 2\alpha u du \propto \alpha\varepsilon^{2/3}(u/\varepsilon^{2/3})d(\nu/\kappa) \propto \alpha\varepsilon^{2/3}(u^{-1/3})\kappa^{-2}d\kappa \propto \alpha\varepsilon^{2/3}\kappa^{-5/3}d\kappa$ that leads to the distribution function $F(\kappa) = \alpha\varepsilon^{2/3}\kappa^{-5/3}$ that is *Kolmogorov-Obukhov* law.

Comparison of invariant *Bernoulli* equation with invariant *Hamilton-Jacobi* equation of classical mechanics [1] resulted in the introduction of the invariant action and quantum mechanics wave function

$$S_\beta(\mathbf{x}, t) = \rho_\beta \Phi_\beta \quad , \quad \Psi_\beta(\mathbf{x}, t) = S'_\beta(\mathbf{x}, t) \tag{6}$$

leading to derivation from *Bernoulli* equation of invariant time-dependent *Schrödinger* equation [3]

$$i\hbar_{o\beta} \frac{\partial \Psi_\beta}{\partial t} + \frac{\hbar_\beta^2}{2m_\beta} \nabla^2 \Psi_\beta - \bar{U}_\beta \Psi_\beta = 0 \tag{7}$$

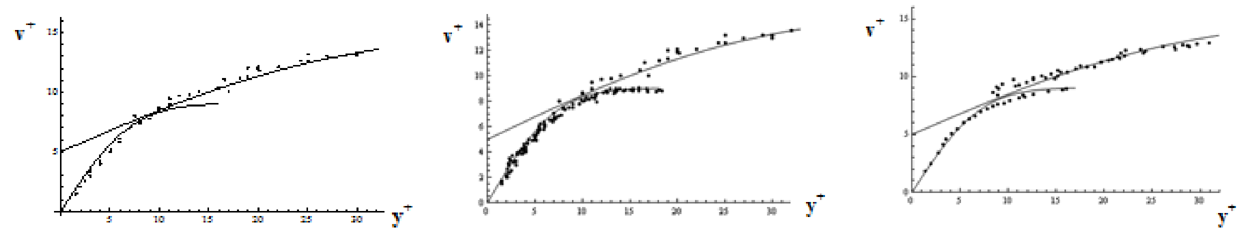
that governs the dynamics of particles from cosmic to tachyon scales (Fig. 1). Since $\bar{E}_\beta = \bar{T}_\beta + \bar{U}_\beta$ [3], time independent *Schrödinger* equation gives the *stationary states* of particles that are trapped within clusters, *de Broglie* wave packets, under the external *potential* acting as *Poincaré* stress [2].

The hierarchies of embedded turbulent flows are most clearly seen in turbulent boundary layer over a flat plate when the solutions of the modified form of equation of motion at scales β +1and β were respectively found to be [3]

$$v_{\beta+1}^+ = 5 + 8(2/\sqrt{\pi})^2 \ \mathrm{erf}(\ y_\beta^+ / 32) \tag{8}$$

$$v_\beta^+ = 8\ (2/\sqrt{\pi})\ \mathrm{erf}(y_{\beta-1}^+ / 8) \tag{9}$$

For example, the solutions in (8)-(9) at β+1 = e and β = c correspond to LED and LCD and their comparisons with experimental data (*Schlichting*, 1968, *Landahl and Mollo-Christensen*, 1992, *Zagarola*, *et al*., 1997) are shown in Fig. 2(a). Similarly, the solutions in (8) -(9) at β+1 = c and β = m correspond to LCD and LMD and at β+1 = m and β = a corresponding to LMD and LAD and their comparisons with experimental data of *Lancien et al.* (2007) and *Meinhart et al*. (1999) are shown in Figs. 2(b) and 2(c), respectively.



(a) (b) (c)

Figure 2. Comparison between the predicted velocity profiles (a) LED-LCD, (b) LCD-LMD, (c) LMD-LAD with experimental data in the literature over 10^8 range of spatial scales [3].

Further implications of the model to the problem of turbulence and quantum mechanics will be discussed.

References

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