ON SOME MATHEMATICAL ROLE OF THE HAMILTONIAN OF VORTICES

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Abstract According to the scenario of Onsager to explain the reason why large scale long lived coherent structures are often observed in two dimensional flows, the mean fields for the equilibrium of a large number of vortices are often considered. In this presentation I would like to explain our recent results concerning the variational structure of the blow-up solutions to the Gel'fand problem, which would give insights on the relation between mean fields of equilibrium vortices and the Hamiltonian of vortices. This is based on the joint works with F. Gladiali (Univ. Sassari), M. Grossi (Univ. Roma, La Sapienza), and T.Suzuki (Osaka Univ.).

THE HAMILTONIAN OF VORTICES

Let Ω be a bounded domain with smooth boundary $\partial\Omega$. We assume that Ω is simply connected for simplicity. A system of N-vortices in Ω is a set $\{(x_j(t), \Gamma_j)\}_{j=1,\cdots,N} (\subset \Omega \times (\boldsymbol{R}\backslash\{0\}))$ that forms a vorticity field $\omega(x,t) = \sum_{j=1}^{N} \Gamma_j \delta_{x_j(t)}$ of two-dimensional incompressible non-viscous fluid in Ω and each point of the system is considered to move according to the following equations:

$$\Gamma_i \frac{dx_i}{dt} = \nabla_i^{\perp} H^{N,\boldsymbol{\Gamma}}(x_1, \cdots, x_N) \left(= \left(\frac{\partial H^{N,\boldsymbol{\Gamma}}}{\partial x_{i,2}}, -\frac{\partial H^{N,\boldsymbol{\Gamma}}}{\partial x_{i,1}} \right) \right), \tag{1}$$

where $x_i = (x_{i,1}, x_{i,2})$ and

$$H^{N,\boldsymbol{\Gamma}}(x_1, \cdots, x_N) = \frac{1}{2} \sum_{j=1}^{N} \Gamma_j^2 R(x_j) + \frac{1}{2} \sum_{1 \le j,k \le N,\, j \ne k} \Gamma_j \Gamma_k G(x_j, x_k).$$

Here $G(x,y)$ is the Green function of $-\Delta$ satisfying $-\Delta G(\cdot, y) = \delta_y$ in Ω and $G(\cdot, y) = 0$ on $\partial\Omega$. $K(x,y) = G(x,y) - \frac{1}{2\pi} \log|x-y|^{-1}$ is the regular part of $G(x,y)$ and $R(x) = K(x,x)$ is the Robin function of $G(x,y)$. The value of $H^{N,\boldsymbol{\Gamma}}(x_1(t), \cdots, x_N(t))$ is preserved under the time evolution and the system is able to be considered as a Hamiltonian system. Therefore $H^{N,\boldsymbol{\Gamma}}$ is called the *Hamiltonian* of N-vortices $\{(x_j(t), \Gamma_j)\}_{j=1,\cdots,N}$.

ON THE THEORY OF ONSAGER FOR THE EQUILIBRIUM VORTICES

Based on this Hamiltonian structure of N-vortices, Onsager tried to explain the reason why large scale long lived coherent structures are often observed in two dimensional flows via the equilibrium statistical mechanics and the concept of *"the negative temperature"* [10]. Indeed Onsager observed that the finiteness of the volume of the phase space Ω^N of N-vortices causes the possibility of the negative temperature $\tilde{\beta} < 0$ states and concluded that this leads *the clusters* of vortices have more possibility to occur. According to this scenario, several researchers tried to see *"the clusters"* in a large number of vortices. Among them here we recall the result of Caglioti et al. [1] and Kiessling [7], which seems to get the conclusion mathematically rigorously using the argument developed by Messer-Sphon [8]. Suppose all the intensities of vortices are equivalent to some constant Γ, which is the case Onsager originally considered. Then the Hamiltonian of N-vortices $H^{N,\boldsymbol{\Gamma}}$ reduces to $\Gamma^2 H^N$, where

$$H^N(x_1, \ldots, x_N) = \frac{1}{2} \sum_{j=1}^{N} R(x_j) + \frac{1}{2} \sum_{1 \le j,k \le N,\, j \ne k} G(x_j, x_k).$$

Using the canonical Gibbs measure μ^N associated to this Hamiltonian $\Gamma^2 H^N$ at the inverse temperature $\tilde{\beta}$, we are able to get the probability (density) of the first vortex observed at $x_1 \in \Omega$ from $\rho^N(x_1) = \int_{\Omega^{N-1}} \mu^N dx_2 \cdots dx_N$. Assuming that the total vorticity is equivalent to 1, that is, $\Gamma = \frac{1}{N}$ and $\tilde{\beta} = \tilde{\beta}_\infty \cdot N$ for some fixed $\tilde{\beta}_\infty \in (-8\pi, +\infty)$, we get ρ satisfying the following equation at the limit of ρ^N as $N \longrightarrow \infty$:

$$\rho(x) = \frac{e^{-\tilde{\beta}_\infty G\rho(x)}}{\int_\Omega e^{-\tilde{\beta}_\infty G\rho(x)} dx}, \tag{2}$$

where $G\rho(x) = \int_\Omega G(x,y)\rho(y)dy$ ([1, Thorem 2.1]). This ρ is called the *mean field* of the equilibrium vortices of one kind and we may see *"the clusters"* of vortices from the structure of the function ρ. Here we should remark that $\tilde{\beta}_\infty$ is able to take a negative value ranges in $(-8\pi, 0)$ from the careful calculations around the singularities of H^N.

We note that the equations similar to (2) are derived by several authors under physically reasonable assumptions and arguments from some systems of vortices, e.g., neutral and two kinds system, that means there exist same numbers of vortices with positive or negative intensities with the same absolute value [5, 6, 11]. See [2] for recent developments of the idea and related topics.

THE GEL'FAND PROBLEM AND THE RESULT

Set $u := -\tilde{\beta}_\infty G\rho$ and $-\tilde{\beta}_\infty / \int_\Omega e^u dx = \lambda$. Then (2) means

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3}$$

which is sometimes called *the Gel'fand problem* (or *the Liouville-Gel'fand problem*) in two space dimensions, which appears in various area of mathematics such as differential geometry, gauge field theories, stationary states of chemotactic motions, and so on. The main object of this presentation is this Gel'fand problem. We are interested in the negative temperature cases $\tilde{\beta}_\infty < 0$, which corresponds to the case $\lambda > 0$. In this range, the following behavior of u_n is observed by Nagasaki and Suzuki [9]: let $\{\lambda_n\}_{n\in\mathbf{N}}$ be a sequence satisfying $\lambda_n \downarrow 0$ and $u_n = u_n(x)$ be a solution to (3) for $\lambda = \lambda_n$. Then it must holds that

$$\lambda_n \int_\Omega e^{u_n}\, dx (= -\tilde{\beta}_\infty) \to 8\pi m \tag{4}$$

for some $m = 0, 1, 2, \cdots, +\infty$ along a sub-sequence. If $m = 0$ the solution converges to 0 uniformly: $\|u_n\|_\infty \to 0$. If $m = +\infty$ there arises the entire blowup in the sense that $\inf_K u_n \to +\infty$ for any $K \Subset \Omega$. If $0 < m < \infty$ the solutions $\{u_n\}$ blow-up at m-points, that is, there is $\mathcal{S} = \{\kappa_1, \cdots, \kappa_m\} \subset \Omega$ composed of m-distinct points such that $u_n \to \sum_{j=1}^m 8\pi\, G(\cdot, \kappa_j)$ in $C^2_{loc}(\overline{\Omega} \setminus \mathscr{S})$, which represents *"the (equi-)clusters"* of vortices since $\rho_n := \frac{-\Delta u_n}{-\tilde{\beta}_\infty} \longrightarrow \sum_{j=1}^m \frac{1}{m}\delta_{\kappa_j}$. It is also proved in [9] that

$$\nabla H^m(\kappa_1, \ldots, \kappa_m) = 0 \tag{5}$$

holds, which suggests that *"the clusters"* of equilibrium vortices locate around the stationary points of m equi-vortices, that is, we may say that *the Hamiltonian has a role to control the locations of "the clusters" of equilibrium vortices*. Concerning the link between H^m and $\{u_n\}$ for m points blow-up cases, we get a deeper one:

Theorem 1 ([4])**.** Suppose the blow-up points $\mathscr{S}$ is a *non-degenerate* critical point of H^m. Then the associated u_n for $n \gg 1$ is a *non-degenerate* critical point of the functional

$$F_{\lambda_n}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda_n \int_\Omega e^u d$$

over the function space $H^1_0(\Omega) = \{u : \Omega \longrightarrow \mathbf{R} \mid \int_\Omega |\nabla u|^2\, dx < \infty,\ u = 0 \text{ on } \partial\Omega\}$.

We note that the Euler-Lagrange equation of the functional $F_{\lambda_n}(u)$ on $H^1_0(\Omega)$ is nothing but the Gel'fand problem (3). In the presentation, we also review our recent results on the calculation of the Morse index of the *functional* F_{λ_n} at u_n for $n \gg 1$ in terms of the Morse index of the *function* H^m at $\mathscr{S}$ [3].

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