SCALING OF PÉCLET AND NUSSELT NUMBERS IN TURBULENT CONVECTION

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Abstract In this paper we present governing equations that could be used for computing the scaling of Péclet number (Pe) and Nusselt number (Nu) in turbulent convection. Under the limiting cases we observe that $\mathrm{Pe} \approx \sqrt{\mathrm{RaPr}}$ for $\mathrm{Re} \gg 1, \mathrm{Pe} \gg 1$; $\mathrm{Pe} \approx \mathrm{RaPr}$ for $\mathrm{Re} \gg 1, \mathrm{Pe} \ll 1$; and $\mathrm{Pe} \approx \mathrm{Ra}^{0.62}$ for $\mathrm{Re} \ll 1, \mathrm{Pe} \gg 1$. We also show that the normalized correlation function between the vertical velocity field and the temperature fluctuations scales as $\mathrm{Ra}^{-0.22}$ for moderate Ra, which would flatten beyond the critical Rayleigh number leading to the "ultimate regime".

The scaling of large-scale properties like Péclet and Nusselt numbers have been modelled using the properties of bulk and boundary layers [3, 4, 6, 10]. In this paper we derive the properties of large-scale quantities by applying scaling arguments on the bulk flow for some regimes of Prandtl and Rayleigh numbers. Our results based on scaling arguments are consistent with the Grossmann-Lohse theory [4], and are in good agreement with experiments and numerical simulations.

One of the generic features observed in all our numerical simulations are the finite amplitude of Fourier modes $\hat{\theta}(0, 0, 2n)$ [7], where the three indices indicate wavenumber components (k_x, k_y, k_z). We observed that

$$\hat{\theta}(0,0,2n) \approx -\frac{\Delta}{2n\pi}. \tag{1}$$

where Δ is the temperature difference between the top and bottom plates, which are separated by a distance d. It is important to note that $\hat{\theta}(0, 0, 2n)$ do not contribute to buoyancy since $\hat{u_z}(0, 0, 2n) = 0$ due to the absence of net vertical mass flux across any horizontal surface. Therefore, the large-scale quantities of the momentum equation are related as

$$c_1 \frac{U_L^2}{d} = \alpha g \theta_{\mathrm{res}} + c_2 \nu \frac{U_L}{d^2}, \tag{2}$$

where where α is the coefficient of thermal expansion, g is gravitational acceleration, ν is the kinematic viscosity, c_1, c_2 are constants, θ_L and U_L are the large scale temperature and velocity fluctuations respectively, and θ_{res} is defined by

$$\theta_{\mathrm{res}}^2 = \theta_L^2 - \sum_n |\hat{\theta}(0,0,2n)|^2. \tag{3}$$

Similarly, the temperature equation of Rayleigh-Bénard convection (RBC) yields

$$c_3 \frac{U_L \theta_L}{d} = \frac{\Delta}{d} U_L + c_4 \kappa \frac{\theta_L}{d^2}, \tag{4}$$

where κ is the thermal diffusivity of the fluid, and c_3, c_4 are constants. The above set of equations succinctly describe the large-scale quantities of RBC. The constants c_1, c_2, c_3, c_4 could be weak functions of Prandtl number, aspect ratio, etc. We can determine the scaling of large-scaling quantities using Eqs. (2,4) once c_1, c_2, c_3, c_4 have been computed from the data from experiments and numerical simulations. Here, we present scaling of U_L and θ_L under limiting cases.

1. $\mathrm{Re} \gg 1$, $\mathrm{Pe} \gg 1$: In this regime, the nonlinear terms of equations. (2,4) are much larger than the diffusive terms, which can be ignored. Therefore, $\theta_L \approx \Delta$ and $\mathrm{Pe} = U_L d/\kappa \approx (\mathrm{RaPr})^{1/2}$. Numerical simulations also reveal that $\theta_L \approx 0.25\Delta$ with $c_3 \approx 4$. Using the fact that $\mathrm{Pe} \approx 0.2(\mathrm{RaPr})^{1/2}$, we compute $c_1 \approx 4.25$. In this case, the thermal and viscous boundary layers are very thin. Hence, bulk scaling provides us good estimate of the large-scale quantities [11].

2. $\mathrm{Re} \gg 1$, $\mathrm{Pe} \ll 1$: Since $\mathrm{Pe} \ll 1$, the term $(\Delta/d)u_L$ matches with the diffusive term in equation (4) thus yielding $\theta_L \approx \mathrm{RaPr}$ and $\mathrm{Pe} \approx \mathrm{RaPr}$. We can also deduce that $\mathrm{Re} \approx \mathrm{Ra}$ for $\mathrm{Pr} = 0$. Here, the thermal boundary layer covers most of the box. Hence, the bulk scaling provides us valuable insights.

3. $\mathrm{Re} \ll 1$, $\mathrm{Pe} \gg 1$: In this case, the nonlinear term of equation (2) is ignored, but the pressure term contributes significantly leading to [8]

$$\nu \nabla^2 \mathbf{u} \approx \alpha g \left[\sum_{\mathbf{k}} |\hat{\theta}(\mathbf{k})|^2 \frac{k_\perp^2}{k^6} \right]^{1/2} \approx \alpha g \Delta \mathrm{Ra}^{-\zeta}. \tag{5}$$

Consequently, $\theta_L \approx 0.28\Delta$ and $\mathrm{Pe} \approx 0.20\mathrm{Ra}^{0.62}$ with $\zeta \approx 0.38$. Pandey *et al.* [8] showed that $c_2 = 0.8$ based on the data from numerical simulation. In this case, the viscous boundary layer spans most of the container, thus making the arguments of bulk scaling quite significant.

We observe that the values deduced from the model are in good agreement with the experimental and numerical results reported earlier, as well as with the predictions of Grossmann and Lohse theory [4]. We are in the process of using the above arguments to compute the Peclét number and θ_L as a function of the Prandtl and Rayleigh numbers. Our work show that the constants c_i's may depend on parameters like Prandtl number, etc. This work is under progress.

The aforementioned scaling of U_L and θ_L are important for understanding the Nusselt number scaling. In the following, we present scaling for moderate Pr.

The Nusselt number, which is a measure of the total heat transfer compared to the conductive heat transfer, is given by [11]

$$\mathrm{Nu} - 1 = \langle u'_z \theta'_{\mathrm{res}} \rangle = C^{u\theta}(\mathrm{RaPr}) \langle u'^2_z \rangle_V^{1/2} \langle \theta'^2_{\mathrm{res}} \rangle_V^{1/2}, \tag{6}$$

where $C_{u\theta}(\mathrm{RaPr})$ is the correlation function between the vertical velocity and the temperature fluctuations. For moderate Prandtl numbers, Verma *et al.* [11] observed that

$$C_{u\theta}(\mathrm{RaPr}) = \left\langle \frac{\langle u'_z \theta'_{\mathrm{res}} \rangle_V}{\langle u'^2_z \rangle_V^{1/2} \langle \theta'^2_{\mathrm{res}} \rangle_V^{1/2}} \right\rangle_t \sim (\mathrm{RaPr})^{-0.2}, \tag{7}$$

at least up to $\mathrm{Ra} \approx 10^8$. Here V and t stand for the volume and temporal averages respectively. From the discussion on the earlier section, $(u'_z)_L \approx (\mathrm{RaPr})^{1/2}$. Therefore,

$$\mathrm{Nu} \approx (\mathrm{RaPr})^{1/2-0.2} \approx (\mathrm{RaPr})^{0.3}, \tag{8}$$

which is observed in experiments and numerical simulations up to $\mathrm{Ra} = 10^{14}$ or so.

In a recent experiment, He *et al.* [5] report an increase in the Nusselt number exponent to approximately 0.38 near $\mathrm{Ra}_{\mathrm{tr}} = 5 \times 10^{14}$, which they attribute to onset of the "ultimate regime" predicted by Kraichnan [6]. Interestingly, the transitional Rayleigh number $\mathrm{Ra}_{\mathrm{tr}}$ corresponds to $\mathrm{Re} \approx \mathrm{Pe} \approx 0.1 \times \sqrt{\mathrm{Ra}_{\mathrm{tr}}} \approx 4 \times 10^6$, which is close to the transitional Reynolds number for the emergence of the turbulent boundary layer in the flow over a flat plate, as well as for the flow past a cylinder. Ahlers *et al.* [1] reported a logarithmic profile for the temperature above $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{tr}}$, which is in general agreement with the logarithmic profile for the velocity in the turbulent boundary layer for the flow past a flat plate. These arguments strongly suggest the birth of a turbulent boundary layer, as well as destruction of the "large-scale circulation", near the plates near $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{tr}}$ [1]. Note that the turbulent boundary layer is expected to appear earlier for rough plates than for smooth ones. This could be the reason for onset of the ultimate regime at lower Ra in the experiments of Chavanne *et al.* [2] and Roche *et al.* [9].

The variation of $C_{u\theta} \sim (\mathrm{RaPr})^{-0.2}$ appears to be a key ingredient for the deviation from Kraichnan's prediction that $\mathrm{Nu} \sim \mathrm{Ra}^{1/2}$ for the "ultimate regime" [11]. Verma *et al.* [12] predict that $C_{u\theta}(\mathrm{RaPr})$ should flatten in the ultimate regime beyond $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{tr}} \approx 10^{15}$.

In this paper we presented scaling of large scale quantities like Peclét and Nusselt numbers. These features demonstrate that bulk properties are very useful in determining characteristics of convection.

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