# COHERENT STRUCTURES IN AXISYMMETRIC TURBULENCE : FROM MICROCANONICAL MEASURES OF THE EULER EQUATIONS TO A VON-KÁRMÁN EXPERIMENT.

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*Abstract* Using an analogy with an Ising-like spin model, we define microcanonical measures for the dynamics of three dimensional (3D) axisymmetric turbulent flow, compute the relevant physical quantities and argue that axisymmetry should induce a large scale organization in turbulent flows. We then comment on the relation between the microcanonical equilibria and the stationnary states observed in a turbulent von-Kármán experiment with axisymmetric geometry.

Statistical mechanics provided extremely powerful tools to study complex dynamical systems in all fields of physics. However it proved extremely difficult to apply classical statistical mechanics ideas to turbulence problems. The main reason is that many statistical mechanics theories relie on equilibrium or close to equilibrium results, based on the microcanonical measures. One of the main phenomena of classical three dimensional (3D) turbulence is the anomalous dissipation : the fact that there exists an energy flux towards small scales that remains finite in the inertial limit of an infinite Reynolds number. This makes the classical 3D turbulence problem an intrinsic non-equilibrium problem. Hence, microcanonical measures have long been thought to be irrelevant for turbulence problems.

A purely equilibrium statistical mechanics approach to 3D turbulence is actually pathological. Indeed, it leads for any finite dimensional approximation to an equipartition spectrum, which has no well defined asymptotic behavior in the limit of an infinite number of degrees of freedom [2]. This phenomena, related to the Rayleigh-Jeans paradox of the equilibrium statistical mechanics of classical field [12], is a sign that an equilibrium approach is bound to fail. This is consistent with the observed phenomena of anomalous dissipation for the 3D Navier-Stokes and suspected equivalent anomalous dissipation phenomena for the 3D Euler equations.

The case of the 2D Euler equations and related Quasi-Geostrophic dynamics, is a remarkable exception to the rule that equilibrium statistical mechanics fail for classical field theories. In this case the existence of new class of invariants (Casimirs), among them enstrophy, leads to a completely different picture. Onsager first anticipated this difference when he studied the statistical mechanics of the point vortex model, which a a class of special solutions to the 2D Euler equations [11, 4]. After the initial works of Robert, Sommeria and Miller in the nineties [9, 14, 15] and subsequent works from then [8, 3, 7, 1], it is now clear that for the 2D Euler equations, microcanonical measures taking into account all invariants exist. These microcanonical measures can be built through finite dimensional approximations. The finite dimensional approximate measure have then a well defined limit, which verifies some large deviations properties (see for instance [13] for a recent simple discussion of this construction). The physics described by this statistical mechanics approach is a self-organization of the flow into a large scale coherent structure corresponding to the most probable macrostate.

The three dimensional axisymmetric Euler equations describe the motion of a perfect three dimensional flow, assumed to be symmetric with respect to rotations around a fixed axis. Such flows have additional Casimir invariants (toroidal Casimirs and generalized helicities, defined below) [6]. By contrast with 2D Euler equations, it has however never been proven that Casimir constraints should prevent an energy cascade towards smaller and smaller scales. It has been stated that the dynamics of such flows lead to predictable large scale structures [10], though. Based on these remarks, the three dimensional axisymmetric Euler equations seem to be an intermediate case between 2D and 3D Euler equations. It is then extremely natural to address the issue of the existence or not of non-trivial microcanonical measures.

In this talk, we define approximate microcanonical measures on the space of finite dimensional approximation of axisymmetric flows, compatible with a formal Liouville theorem. As the fixed invariant subspace of the phase space is not bounded, we also have to consider an artificial cutoff $M$ on the accessible vorticity values. From these approximate microcanonical measures, we compute the probability distribution of the orthodial vorticity field and of the orthoradial velocity field. The microcanonical measure of the 3D axisymmetric equations is defined as a weak limit of sequences of those finite dimensional approximate microcanonical measures, when the cutoff $M$ goes to infinity. More heuristically, we will show that the Euler equations can be mapped onto an Ising-like model whose thermodynamic limit corresponds to a microcanonical measure of the Euler equations. We prove that this limit exists and that it describes non-trivial flow structures.

A very interesting yet quite subtle question is the interest of such a microcanonical measure for experimental 3D flows in an axisymmetric geometry. We observe that data obtained in a turbulent von-Kármán experiment [10, 5] seem to be compatible with the measures obtained in the limit when the cutoff $M$ goes to zero, whereas the actual microcanonical measures are obtained in the limit $M \to \infty$ (see Figure 1).

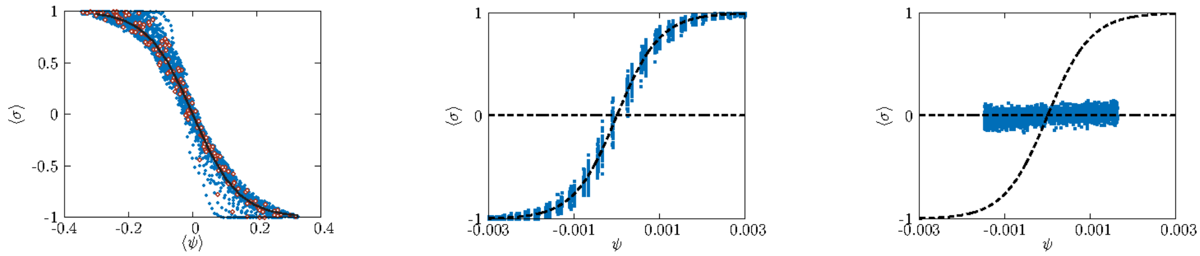


**Figure 1.** Typical relation between the mean of a discretized toroidal field $\sigma$ and the stream function $\psi$ obtained in a von-Kármán experiment (left), and in Monte-Carlo simulations of the microcanonical measure performed for a vorticity cut-off that is a vanishing (middle) or large (right).

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