SUB-GRID EFFECTS OF THE VOIGT VISCOELASTIC REGULARIZATION

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Abstract In this work we discuss the spectral signature of the Navier-Stokes-Voigt (NSV) viscoelastic fluid flows by employing numerical simulations of a singular dyadic shell model, and comparing it with former works, where 2D DNS and the Sabra shell model were used instead. Our results clearly show that as the relaxation time is increased above a threshold, the inertial range is reduced, conserving part of the large-scale statistics. We also show that the additional elastic term regularizes the singular dyadic model, which is the main reason behind this reduction of degrees of freedom. The results of this work aim at proposing some viscoelastic modifications of the Navier-Stokes equations as sub-grid models.

THE VOIGT SUB-GRID MODEL

Several features of Kolmogorov theory of turbulence were independently discovered by Lars Onsager, who proposed that weak solutions of Euler equations could develop singularities in finite time, which can be the ultimate source for the power-law scaling of the structure functions, see [3]. Another major feature of turbulence theory discussed both by Kolmogorov and Onsager is the notion of dissipative range of scales, below which the effects of viscosity are prevalent.

This range, despite of carrying only a small fraction of the total kinetic energy of the flow, represents the vast majority of scales involved in direct numerical simulations. The main objective of sub-grid models is to accurately reproduce the large-scale details of turbulent flows without the need to compute too much of the dissipative range, which is the main source of computational costs. The essence of these models is to model the energy transfer between large and small scales through a modified inertial or diffusive non-linear term.

Understanding the intricate details of the interplay between the inertial range and the dissipation range is crucial not only for numerical analysis purposes, but also for understanding the still very mysterious physics of viscoelastic flows, see [6], where the energy cascade scenario is modified with the additional relaxation time scales of the polymeric structure.

A long-time tradition of analysing viscoelastic materials from a pure macroscopic point of view comes from the times of Kelvin, Maxwell and Voigt, see [6]. It amounts to consider appropriate modifications of the Cauchy stress tensor, which ultimately yields its viscoelastic nature.

In the pure macroscopic approach, the main rheological properties of a viscoelastic flow is defined by its constitutive law, and in particular, by its Cauchy stress tensor. For Kelvin-Voigt materials, it is given by $\sigma = -p\mathbf{I}+2\mu\mathbf{D}+2\beta\frac{\partial \mathbf{D}}{\partial t}$, where $\mathbf{D}$ is the symmetric component of the velocity gradient, and $\frac{\partial \mathbf{D}}{\partial t}$ is the stretch tensor. The coefficient μ is the shear viscosity and β is the shear modulus of the elastic component.

Considering this class of stress tensor for incompressible fluids, and substituting it in the linear momentum balance, gives rise to the equations of motion for the Navier-Stokes-Voigt fluids:

$$\begin{cases} \partial_t(\mathbf{u}-\alpha^2\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \quad \mathbf{x}\in\Omega, \\ \nabla\cdot\mathbf{u} = 0, \quad \mathbf{x}\in\Omega, \\ \mathbf{u}(\mathbf{x},t) = 0 \quad \mathbf{x}\in\partial\Omega. \end{cases} \tag{1}$$

The Voigt regularisation is known to prevent formation of singularities for both incompressible Euler and Navier-Stokes equations, see [1]. This blow-up prevention mechanism steams from a change of time-scales related to the very small length scales. This behaviour is actually observed in a variety of viscoelastic flows. This hybrid time-scale behaviour besides regularizing the solutions is also responsible for reducing the degrees of freedom of the flow.

Indeed, in a recent work, [4], the authors test the Voigt regularization in benchmark problems of the two-dimensional Navier-Stokes and MHD equations. Their results show the efficacy of this regularization in numerical simulations performed in a coarse grid in domains with physical boundaries, building up more evidence that this model can be used as a sub-grid model. Numerical simulations of the full three-dimensional NSV equations, however, is certainly an extremely demanding task for very high Reynolds number flows, just like it is for the Navier-Stokes equations.

SHELL MODEL SIMULATIONS OF THE VOIGT REGULARIZATION

In order to get insight into the turbulence problem, several simplified models have been developed in the past. One of the most studied models are the shell models of turbulence, which are phenomenological models inspired by a severe truncation and averaging of the equations of motion in Fourier space. Shell models of turbulence have a long history in the fluid mechanics community, since the pioneering works of Obukhov. These models possess conserved quantities compatible with the unforced 3D incompressible Euler equations, and their main weakness is the absence of geometrical informations in the real space. Nevertheless, it is very useful to test simple ideas concerning the statistical spectral dynamics of turbulent flows. In [5], the authors studied the Voigt model via the Sabra shell model. The results showed a secondary power-law with a reduced inertial range.

A large set of phenomenological assumptions in turbulence lead naturally to a large set of types of shell models. Recently, several works analysed the so-called dyadic model, which is known to produce both power-law scaling and Onsager-like singularities when viscosity is set to zero, see [2]. The viscous dyadic model is the following system of non-linear ordinary differential equations:

$$\frac{du_n}{dt} + \nu\lambda^{2n}u_n - \lambda^n u_{n-1}^2 + \lambda^{n+1}u_n u_{n+1} = f_n, \tag{2}$$

for $n = 1, 2, 3, \ldots$, with the boundary condition $u_0 = 0$, and where $\lambda > 1$ is the shell spacing parameter. The dyadic model is based on a Littlewood-Paley decomposition of the Euler equations, where each ODE represents a wavelet coefficient describing the behaviour of the velocity localized to a certain frequency shell.

The inviscid dyadic model is known to produce singularities which are compatible with Onsager's theory. It possesses an inviscid dissipation mechanism provided by the singularity, similarly to the Burgers' equations. The main reason for this dissipation is the monotonic transfer of energy to the high modes provided by the non-linear term.

We show that the Voigt term does regularize the singular inviscid dyadic model, and that dissipation range oscillations are indeed tamed by the additional regularization, with a significant reduction of the inertial range's size. This reduction nonetheless does not interfere with the larger scales of motion, and this is the reason we argue for the NSV to be tested as a possible sub-grid model in more extensive simulations. We also present several relations between the dissipative cut-off scale and the relaxation time parameter, α^2/ν.

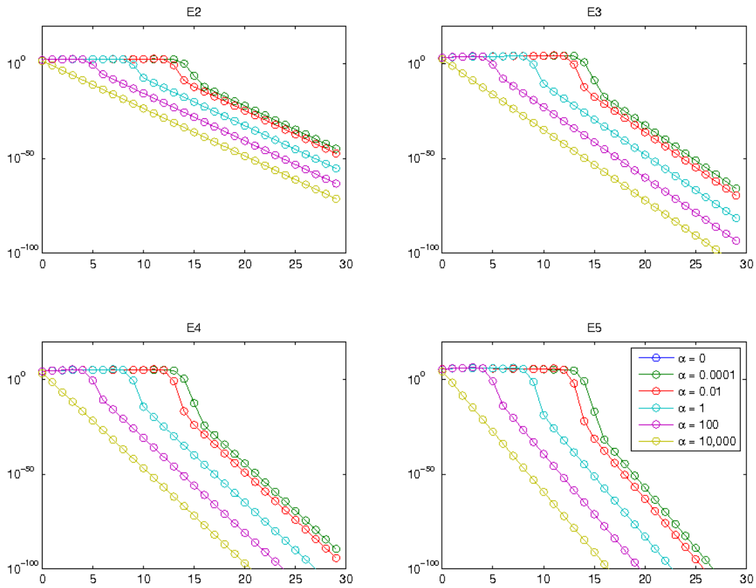

Figure 1. Semi-log plot of compensated structure functions, $S_n(k) \cdot k^{n/3}$. Viscosity = $\nu = 10^{-12}$ and several values of α.

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