FOKKER-PLANCK DESCRIPTION OF THE INVERSE CASCADE IN TWO-DIMENSIONAL TURBULENCE

Oliver Kamps[1] & Michel Voßkuhle [2]
[1] *Center for Nonlinear Science, University of Münster, Münster, Germany*
[2] *Laboratoire de Physique, École normale supérieure de Lyon, Lyon, France*

Abstract In many approaches the mathematical description of fully developed turbulence relies on the statistical properties of the longitudinal velocity increments $\xi(r) = U(x+r) - U(x)$. In [1] the increment statistics is described as a Markov process in scale, leading to a Fokker-Planck description of the probability density functions (PDFs) for the velocity increments. Here we want to extend this description to the inverse energy cascade in two-dimensional turbulence. The central question is whether the velocity field of the inverse cascade can be modeled as a Markov process in scale similar to the three-dimensional case. By estimating the coefficients of the Fokker-Planck equation we are able to discuss the role of intermittency and differences to three-dimensional flows

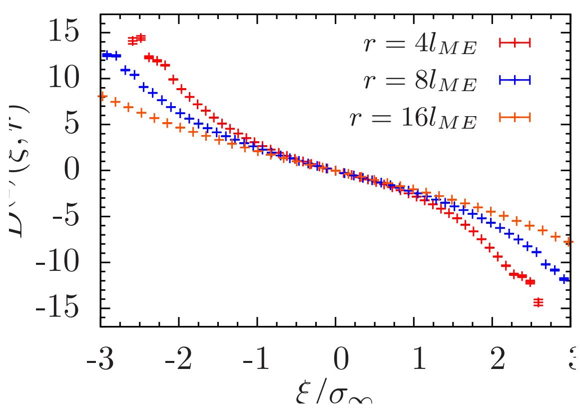
One of the main challenges for the statistical description of fully developed turbulence is to find a compact mathematical formulation that includes all relevant phenomena of the flow field. Since the pioneering works of Richardson and Kolmogorov the description of the energy transport in scale in the velocity field of homogeneous isotropic turbulence has been identified as one of the most important phenomena in turbulence. The mathematical formulation of a scale dependent statistical theory relies in most approaches on the statistical properties of the longitudinal velocity increments $\xi(r) = U(x+r) - U(x)$. Especially phenomenological approaches based on dimensional analysis like the K41 theory or the multifractal model have been developed to describe the scale dependent increment statistics [3].

A promising approach for a compact description of the cascade was proposed by Friedrich and Peinke [1, 2]. In the framework of their theory the increment statistics is described as a Markov process in scale, leading to a Fokker-Planck description of the probability density functions (PDFs) for the velocity increments. The universality of this approach was tested for different kinds of three-dimensional flows like inhomogeneous [8] or fractal grid generated turbulence [7]. Also for the transition of a flow from a vortex street to fully developed turbulence in a cylinder wake the flow can be described as a Markov process in scale [4]. The common feature of all investigated flows is, that the Markov length is in the same order of magnitude as the Taylor length scale [5].

Here we want to extend the test for the universality of the Markov description by analyzing data from numerical simulations of the inverse energy cascade in two-dimensional turbulence. To this end we simulate a two dimensional flow by means of the vorticity-equation

$$\frac{\partial}{\partial t}\boldsymbol{\omega}(\boldsymbol{x},t) = \nabla \times [\boldsymbol{u}(\boldsymbol{x},t) \times \boldsymbol{\omega}(\boldsymbol{x},t)] + \nu\Delta^n \boldsymbol{\omega}(\boldsymbol{x},t) + \gamma\Delta^{-m}\boldsymbol{\omega}(\boldsymbol{x},t) + \nabla \times \boldsymbol{f}(\boldsymbol{x},t)\,. \tag{1}$$

Here $\boldsymbol{u}(\boldsymbol{x},t)$ is the velocity and $\boldsymbol{\omega}(\boldsymbol{x},t) = \nabla \times \boldsymbol{u}(\boldsymbol{x},t)$ is the vorticity. The equation includes the forcing term $\boldsymbol{f}(\boldsymbol{x},t)$ that injects energy into the small scales, a hyper-viscous term of order n and a large scale friction term of order m to



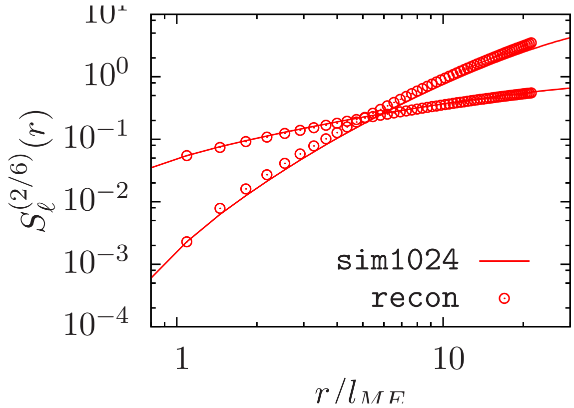


Figure 1. Left: Drift-coefficient for different scales. Right: Reconstruction (denoted as recon) of the second and sixth order longitudinal structure functions from the numerically estimated drift- and diffusion-coefficients in comparison with the directly calculated structure functions.

remove the energy from large scales, thus preserving a statistically stationary state. For the data shown in this work, we chose a hyper-viscous term of order eight and a hyper-friction term of order one.
From the numerical data we demonstarte that the N-point PDF $p(\xi_1, \xi_2, \ldots, \xi_N)$ ($\xi_i := \xi_{r_i}$) for the increments on different scales can be written as

$$p(\xi_1, \xi_2, \ldots, \xi_N) = p(\xi_1|\xi_2)p(\xi_2|\xi_3)\ldots p(\xi_{N-2}|\xi_{N-1})p(\xi_N) \tag{2}$$

for $r_{i+1} - r_i \geq l_{ME}$ with $l_{ME} \approx r_\lambda$ where l_{ME} is the Markov-Einstein length [5] and r_λ is the Taylor length scale. To derive an evolution equation for the transition PDFs $p(\xi_i|\xi_{i-1})$ one can use the Kramers-Moyal expansion [6]. Under the condition that the third-order term of the expansion vanishes [6] this evolution equation is the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(\xi_j|\xi_k, u_i) = -\frac{\partial}{\partial \xi_j}\left[D^{(1)}(\xi_j, r_j, u_i)p(\xi_j|\xi_k, u_i)\right] + \frac{\partial^2}{\partial \xi_j^2}\left[D^{(2)}(\xi_j, r_j, u_i)p(\xi_j|\xi_k, u_i)\right] \tag{3}$$

with the drift-coefficient $D^{(1)}$ and the diffusion-coefficient $D^{(2)}$ defined as

$$D^{(n)}(\xi_j, r_j) = \frac{1}{n!}\lim_{\Delta r \to 0}\frac{1}{\Delta r}M^{(n)}(\xi_j, r_j). \tag{4}$$

These coefficients can be estimated from the numerical data (see left part of fig. 1). Given $D^{(1)}$ and $D^{(2)}$the whole N-point PDF or e.g. the structure functions can be reconstructed via the Fokker-Planck equation (see right part of fig. 1).

In our contribution we will develop the method in detail and discuss the results for various choices of parameters for the two dimensional flow. Special attention will be paid to an elaborate comparison between results from two- and three-dimensional turbulence and the role of intermittency for the structure of the drift- and diffusion coefficients. We also discuss the possibility to extend the Fokker-Planck approach to scales smaller than the Taylor scale.

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