ON THE STRENGTH OF THE NONLINEARITY IN ISOTROPIC TURBULENCE

Robert Rubinstein[1], Wouter J.T. Bos[2]
[1]*Newport News, VA, USA*
[2]*LMFA, CNRS, Ecole Centrale de Lyon, Université de Lyon, Ecully, France*

Abstract In Navier-Stokes turbulence, the variance of the nonlinear term is smaller than it would be in a Gaussian random field with the same energy distribution. Analysis of the variance spectrum shows that this 'depletion of nonlinearity' is evidence of statistical dependence among the Fourier modes, even of modes corresponding to very large and very small scales. We apply the Direct Interaction Approximation to evaluate the scaling of the variance of the nonlinearity at high Reynolds number and show that it is dominated by sweeping effects. The variance reduces to its Gaussian value in thermal equilibrium; we show the growth and decay of the cumulant contribution to the variance of the nonlinearity in transient evolution of the spectrally truncated Euler equations from an assumed Gaussian initial state to a Gaussian final state of thermal equilibrium.

INTRODUCTION

In turbulent flows the variance of the nonlinear term is smaller than it would be in a Gaussian random field with the same correlation function [3]. This observation was termed the 'depletion of nonlinearity.' Various mechanisms that would be absent in a Gaussian random field, such as the preferential alignment of velocity and vorticity ('Beltramization'), were proposed to explain it. But the Gaussian distribution itself is not so crucial; the critical point is the statistical independence of the Fourier modes in a Gaussian random field with uncorrelated amplitudes. From this viewpoint, the observed variance reduction is evidence of statistical dependence of the Fourier modes. This dependence might be considered in turn a signature of some type of self-organization of the turbulent flow, although the exact nature of this process (structure formation is one possibility) cannot be determined by this type of statistical analysis alone.

In Chen *et al.* [2], the spectrum of the variance of the nonlinear term is analyzed in order to reveal the scale-dependence of the depletion of nonlinearity. This spectrum $w(k)$, defined so that $\int_0^\infty w(k)dk = \langle |\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p|^2\rangle$. can be decomposed into a Gaussian contribution and a cumulant, $w(k) = w_G(k) + w_C(k)$, where, in the notation of [2]

$$w_G(k) = k^3 \int_\Delta a(k,p,q)E(p)E(q)\frac{dp\,dq}{pq} \tag{1}$$

The cumulant was computed in this reference using the Direct Interaction Approximation (DIA); in order to give a more concise analytical representation, we use their results, but assume exponential two-time dependence, to obtain

$$\begin{aligned} w_C(k) = \tfrac{1}{2}\int_\Delta \frac{dp'dq'}{p'q'}\int_\Delta \frac{dp\,dq}{pq} b(k,p,q)b(k,p',q')k^4p^2p'^2E(q)E(q')\times \\ \Theta(k;p,q,p',q')\left[2\frac{E(k)}{k} - \frac{E(p)}{p} - \frac{E(p')}{p'}\right]\frac{dp'dq'}{p'q'}\frac{dp\,dq}{pq} \end{aligned} \tag{2}$$

where $\Theta(k;p,q,p',q') = \Theta(k;p',q',p,q)$ is the result of a double time integration. The notation is again that of [2], but we introduced K41-compatible time scales in computing Θ. The details follow the computations in [1].

PROPERTIES OF THE NONLINEARITY SPECTRUM

Dimensional analysis suggests the inertial range scaling $w(k) \sim \epsilon^{4/3}k^{-1/3}$, but the integrals for $w_G(k)$ and $w_C(k)$ both diverge at low wavenumbers on a Kolmogorov spectrum; consequently, dimensional scaling is replaced by $w(k) \sim \epsilon^{2/3}k_0^{-2/3}\epsilon^{2/3}k^{1/3}$ where k_0 is a large-scale cutoff. Alternatively, in terms of the total velocity variance (twice the kinetic energy) U^2, $w(k) \sim U^2\epsilon^{2/3}k^{1/3}$. Thus, the nonlinearity spectrum is dominated by sweeping effects. This conclusion is expected, since the variance of the nonlinearity is also the variance of the Eulerian acceleration, $\langle(\dot{u})^2\rangle$ which is well-known from Kraichnan's work to be sweep-dominated. We remark that the connection to the Eulerian acceleration variance suggests another interpretation of the depletion of nonlinearity as increased Eulerian time coherence of the turbulent field compared to a Gaussian counterpart.

In Figure 1 (left), we show the nonlinearity spectrum, obtained by numerical integration, normalized by 'sweeping-variables', $\tilde{w}(k) = w(k)/(U^2\epsilon^{3/4}\nu^{-1/4})$. We observe that the spectra collapse at high wavenumbers. To a good approximation, the spectra scale as $w(k) \sim U^2\epsilon^{2/3}k^{1/3}f(k\eta)$ with $f(k\eta)$ a function which tends to a constant in the inertial range and which rapidly decays in the dissipation range. A clear inertial range power-law scaling proportional to $k^{1/3}$ appears, however, only at relatively high Reynolds number. For moderate and low Reynolds numbers the power law is

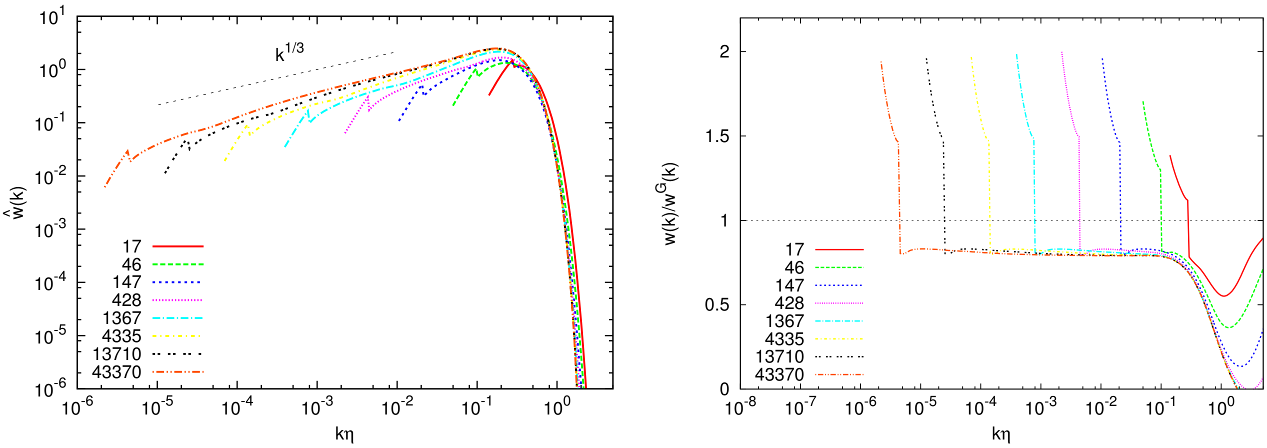

Figure 1. Left: Spectrum of the nonlinear term for different Reynolds numbers. Right: the same spectra, normalized by its Gaussian estimate.

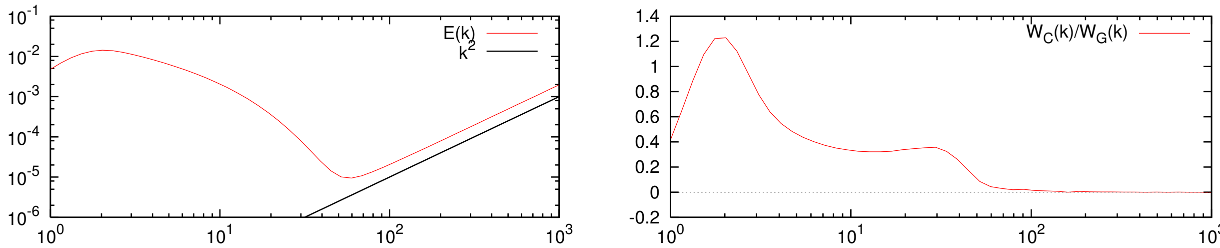

Figure 2. Left: Energy spectrum for truncated Euler flow. Right: spectrum of the nonlinear term, normalized by the Gaussian contribution.

steeper than $1/3$. The nonlocal character of sweeping might explain the slow convergence to the expected asymptotic inertial range scaling.

In Figure 1 (right), we plot the ratio of the nonlinearity spectrum to its Gaussian estimate, $w(k)/w_G(k)$. The nonlinearity spectrum exceeds its Gaussian estimate at large scales, however since these scales are close to the forcing scale and are consequently non-universal, it is difficult to make any conclusion from this observation. The ratio $w(k)/w_G(k)$ is constant in the inertial range but is reduced well below its inertial range value in the dissipation range. The significance of this observation is a topic for future research. An explanation for the exact constant value of $w(k)/w_G(k)$ is currently under investigation.

It is immediately evident from Eq. (2) that w_C vanishes for an equipartition spectrum $E(k) \propto k^2$. The transient evolution of the cumulant contribution $w_C(k)$ in the spectrally truncated Euler equations is therefore of interest: for suitable initial conditions, an equipartitioned 'thermalized tail' forms at wavenumbers below the spectral cutoff and propagates to smaller wavenumbers until it fills the entire spectrum and a steady state is reached [4]. If the initial condition is Gaussian, then the cumulant vanishes initially, grows as nonlinearity develops, and ultimately returns to zero at equipartition. A typical intermediate state is shown in Figure 2 which exhibit the thermalized tail of the energy spectrum and the ratio $w(k)/w_G(k)$; note that it vanishes at the wavenumbers corresponding to the thermalized tail.

Further details concerning the dependence of large and small scales suggested by this analysis and concerning the transient evolution of the truncated Euler equations will be presented in the final paper.

References

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