ENERGY AND GEOMETRY OF A TANGLE OF VORTEX FILAMENTS

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Abstract By performing a numerical experiment, we show that the energy of a superfluid turbulent vortex tangle is not proportional to its length, as usually assumed in the literature. We discuss the consequence in the interpretation of turbulence decay experiments.

INTRODUCTION

At temperatures T close to absolute zero, the fundamental properties of matter become more apparent. In particular, turbulence acquires a simpler form. Turbulence in superfluid helium, or quantum turbulence[1], consists of an apparently disordered tangle of reconnecting vortex filaments moving in an inviscid fluid. Each vortex filament carries one quantum of circulation $\kappa = h/m$ (where h is Planck's constant and m is the mass of one helium atom). In other words,

$$\oint_C \mathbf{v}_s \cdot \mathbf{dr} = \kappa, \tag{1}$$

where C is any path of integration around the axis of the vortex filament and $\mathbf{v}_s$ is the superfluid velocity. The vortex core radius, constrained by quantum mechanics to the fixed value $\xi \approx 10^{-8}$ cm, is much smaller than the average distance between filaments $\ell \approx 10^{-4}$ to 10^{-2} cm in typical experiments; in turn, ℓ is much smaller than the typical system size $D \approx 1$ cm. Because of this separation of scales, $\xi \ll \ell \ll D$, it is possible to model quantum turbulence using the classical theory of thin-cored vortex filaments expressed by the Biot-Savart law [2]. If we raise the temperatures above $T > 1$ K, the normal fluid fraction of liquid helium, consisting of thermal excitations, is not negligible any longer and exerts a small friction force on the vortex filaments; this effect must be added to the equation of motion.

The question which we address in this work is the relation between length and energy of the vortex filaments. What is measured in the experiments using various techniques (second sound, ion trapping, Andreev scattering) is the vortex line density $L = \Lambda/V$, defined as vortex length Λ per unit volume V, from which one infers $\ell \approx L^{-1/2}$. However, the quantity of the greatest physical interest is not the length of the filaments but rather the kinetic energy E of the fluid which rotates around them. In the literature[1, 3], it is common to assume that energy and length are proportional,

$$E = \epsilon \Lambda, \tag{2}$$

where the energy per unit length

$$\epsilon = \frac{E}{\Lambda} \approx \frac{\rho_s \kappa^2}{4\pi} \ln{(\ell/\xi)}, \tag{3}$$

is estimated from a straight vortex filament (here ρ_s is the superfluid density). Our aim is to show that this simple relation between between E and Λ is not correct, particularly in the low temperature regime of a pure superfluid.

RESULTS

We represent vortex filaments as space curves $\mathbf{s}(t)$ which move according to [4]

$$d\mathbf{s}/dt = \mathbf{v}_s + \alpha \mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_s) - \alpha' \mathbf{s}' \times (\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_s)), \tag{4}$$

where $\mathbf{v}_n$ is the imposed normal fluid velocity, $\mathbf{s}'$ is the unit tangent vector to the filament at the point $\mathbf{s}$, and α, α' are known friction coefficients, the self-induced velocity of the vortex filaments, $\mathbf{v}_s$, is given by the classical Biot-Savart law [2, 4]

$$\mathbf{v}_s(\mathbf{s}) = -\frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{(\mathbf{s} - \mathbf{r})}{|\mathbf{s} - \mathbf{r}|^3} \times \mathbf{dr}, \tag{5}$$

where the line integral extends over the entire vortex configuration. We perform our calculations in a periodic cube of size D and volume $V = D^3$. Starting from an arbitrary initial condition (for example few vortex rings), we create a turbulent vortex tangle at nonzero temperature driven by an imposed turbulent normal fluid velocity $\mathbf{v}_n$ [5, 6]. After an initial transient ($0 < t < 0.5$ s in Fig. 1), a statistically steady state is achieved which is independent of the initial condition, in which the vortex line density L fluctuates about a well-defined average value. An example of such a tangle is shown at the left in Fig. 2. We determine the superfluid kinetic energy E and verify that the distribution of kinetic energy over the length scales is consistent with the classical Kolmogorov scaling $k^{-5/3}$ which is observed in ordinary turbulence, where k is the magnitude of the 3-dimensional wavenumber.

At time $t = 1$ s, in the statistically steady state regime $0.5 < t < 1$ s, we perform a numerical experiment: we suddenly "cool" the system to $T = 0$ by setting $\alpha = \alpha' = 0$. Fig. 2 (right) shows the vortex tangle at $t > 1$ s. The change is remarkable: without friction, Kelvin waves (helical displacements of the vortex cores) are not damped, and the vortex filaments become very wiggly; the effect is quantified by the change of the curvature histogram. The main finding is that, although the energy of the vortex tangle does not change at $t = 1$ s, the vortex length increases by about a third (see Fig. 1), in disagreement with Eq. 2. At larger times $t > 1.1$ s, the tangle decays due to numerical dissipation which in our model plays the role of phonon emission [9].

We conclude that the energy of a vortex tangle is not related to its length as predicted by Eq. 2. The presence of Kelvin waves increases the length at no energy cost, as their contributions to the velocity field tend to cancel out. The effect has implications for the interpretation of experiments of turbulence decay: from the observed decrease of the length one infers the decrease of energy [7, 8], and compares it to energy decay in ordinary turbulence.

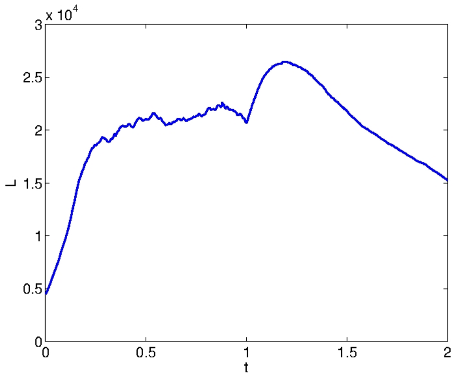

Figure 1. Vortex line density L (cm^{-2}) vs time t (s). Note the sudden increase of vortex length at $t = 1$ s.

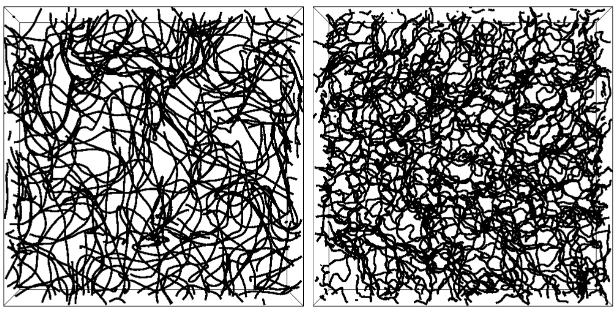

Figure 2. Vortex tangle at nonzero temperature just before $t = 1$ s, and at (right) zero temperature at after $t = 1$ s.

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