RENORMALIZATION OF THE FRAGMENTATION EQUATION: EXACT SELF-SIMILAR SOLUTIONS AND TURBULENT CASCADES

Vladimir Saveliev[1], Michael Gorokhovski[2]
[1]*Institute of Ionosphere, National Center of Space Research and Technology, 050020, Almaty, Kazakhstan*
[2]*Laboratoire de Mécanique des Fluids et d'Acoustique, CNRS - Ecole Centrale de Lyon – INSA Lyon – Université Claude Bernard Lyon 1, 36 Avenue Guy de Collongue, 69134 Ecully Cedex, France*

Abstract Using an approach developed earlier for renormalization of the Boltzmann collision integral (V.L. Saveliev and K.Nanbu, Phys. Rev. E 65, 051205, 2002), we derive an exact divergence form for the fragmentation operator. Then we reduce the fragmentation equation to the continuity equation in size-space, with the flux given explicitly. This allows us to obtain new self-similar solutions, and to find the new integral of motion for these solutions (we call it the bare flux). We show how these solutions can be applied as a description of cascade processes in three and two dimensional turbulence. We also suggested an empirical cascade model of impact fragmentation of brittle materials..

RENORMALIZATION OF FRAGMENTATION EQUATION

The fragmentation process consists of the generation of random fragments (or particles) by successive breaking[1]. The process occurs in numerous physical phenomena and engineering applications. One approach that is often used to model these processes is the fragmentation under scaling symmetry model. The scaling symmetry means that at each fragmentation step, a parent particle splits into daughter particles with a partition probability independent of the particle size, *i.e.* the particle size r is modified in a step by step manner by multiplication by a random multiplier $r \Rightarrow \alpha r$; $0 \le \alpha \le 1$. We will consider the case when fragmentation rate ν is a power function on size. The fragmentation events result in an increase in the total number of particles over time, whilst the total mass of particles is conserved. Therefore instead of the number distribution function, the mass distribution function $f(r)$ is usually used. The norm of this function is conserved. Thus the fragmentation equation for $f(r)$ takes the following form:

$$\begin{gathered}\frac{\partial f(r)}{\partial t} = \left(\hat{I}_+ - 1\right)\nu(r) f(r), \quad \nu(r) = c r^{\mu}, \\ \hat{I}_+ = \int_0^1 d\alpha\, q(\alpha) e^{-\ln\alpha\left(r\frac{\partial}{\partial r}+1\right)}, \quad e^{-\ln\alpha\left(r\frac{\partial}{\partial r}+1\right)} f(r) = \frac{1}{\alpha} f\left(\frac{r}{\alpha}\right), \quad \int_0^1 d\alpha\, q(\alpha) = 1.\end{gathered} \tag{1}$$

Using an approach developed earlier for renormalization of the Boltzmann collision integral [2], we derive an exact divergence form for the fragmentation operator. Then we reduce the fragmentation equation to the continuity equation in size-space, with the flux given explicitly[1]:

$$\begin{gathered}\frac{\partial}{\partial t} f = \frac{\partial}{\partial r} r\left\langle -\ln\alpha\right\rangle \hat{I}_+^{(1)} \nu f, \\ \hat{I}_+^{(1)} = \int_0^1 d\alpha\, q_1(\alpha) e^{-(\ln\alpha)\hat{\partial}}, \quad q_1(\alpha) = \frac{1}{\left\langle -\ln\alpha\right\rangle}\int_0^1 q(\alpha\beta)\, d\beta\end{gathered}. \tag{2}$$

If the fragmentation spectrum $q(\alpha)$ is a power function $q(\alpha) = (\gamma+1)\alpha^{\gamma}$ with $\gamma > -1$, then equations can be simplified significantly because of the property:

$$q_1(\alpha) = q(\alpha), \quad \hat{I}_+^{(1)} = \hat{I}_+,. \tag{3}$$

and exact practically important solutions to fragmentation equation can be obtained. The renormalized fragmentation equation contains an explicit expression for the mass-flux. This allows us to derive a new integral of motion j_0 for self-similar solutions. This integral is referred to as the bare flux.

INTERMEDIATE ASYMPTOTICS: SELF-SIMILAR SOLUTIONS TO THE RENORMALIZED FRAGMENTATION EQUATION

Amongst all the solutions, self-similar solutions are of special interest; these solutions are essentially the intermediate long-time asymptotics. For negative values of μ, the self–similar solutions can be expressed by confluent hypergeometric function in the following form:

$$f\left(r,t\right)=\frac{\left(1+\gamma\right)\left|j_0\right|\tau_0}{r}\alpha^{-1}\Phi\left(1,1+\alpha,-\nu t\right),\ \nu=cr^{\mu},\ \alpha=-\frac{1+\gamma}{\mu},$$
$$j=-\frac{\left|j_0\right|\tau_0}{t}\left(1-\Phi\left(1,1+\alpha,-\nu t\right)\right),\ j<0, \tag{4}$$
$$j(r=0)=-\frac{\left|j_0\right|\tau_0}{t},\qquad 0<t<\infty.$$

This solution describes the fragmentation process, subjected to a second order phase transition at zero time, $t=0$.

APPLICATION TO TURBULENT CASCADE

Solution (4) has an interesting application to the description of the turbulent cascade process. In inviscid flows, mechanical energy is conserved, so instead of mass-distribution in size space, we could consider the distribution of specific turbulent energy in the size-space of turbulent eddies undergoing a turbulent cascade. The specific energy distribution function and the specific energy flux have the following dimensions: $\left[f\right]=v^2/r$ and $\left[j\right]=v^3/r=r^2/t^3$, respectively. The eddy-fragmentation process, described by solution (4), is characterized by two dimensional constants j_0 and c. For maximum symmetry of this solution, we assume that constant c can be expressed in terms of j_0 as $c=\left|j_0\right|^x$. This gives $x=1/3,\quad \mu=-2/3,\quad \alpha=\left(3/2\right)\left(1+\gamma\right)$. In infinitely high Reynolds number turbulence, the flux of specific turbulent energy at zero size has the standard notation ε, Solution (4) for integer values $\alpha=n,\ n=1,2,3,...;$ can be expressed in terms of elementary functions. Using $x=1/3$, $\mu=-2/3$; $\gamma=1$; $\alpha=3$, the expressions (4) can be applied to the problem of decaying turbulence:

$$\mu=-\frac{2}{3}<0,\ \gamma=1,\ \alpha=3,\ -\infty<t<\infty,$$
$$f\left(r,t\right)=\frac{2\varepsilon\left(t\right)}{r\nu}\left(1-\frac{2}{\nu t}+\frac{2}{\left(\nu t\right)^2}-\frac{2e^{-\nu t}}{\left(\nu t\right)^2}\right),\qquad \nu\left(r\right)=\varepsilon_0^{\frac{1}{3}}r^{-\frac{2}{3}},\ \ \varepsilon=-j\left(r=0,t>0\right),$$
$$j\left(r,t\right)=-\varepsilon\left(t\right)\left[1-\frac{3}{\nu t}\left(1-\frac{2}{\nu t}+\frac{2}{\left(\nu t\right)^2}-\frac{2e^{-\nu t}}{\left(\nu t\right)^2}\right)\right],\ j<0, \tag{5}$$
$$\varepsilon\left(t\right)=\frac{\varepsilon_0\tau_0}{t},\ \ \varepsilon_0=\varepsilon\left(\tau_0\right)=\left|j_0\right|.$$

Under the time translation $t\to t+\tau_0$ in eq. (5), and setting $\tau_0\to\infty$, the following stationary solution takes place:

$$f\left(r\right)=2\,\varepsilon_0^{2/3}r^{-1/3},\qquad j=-\varepsilon_0<0\,. \tag{6}$$

Then the distribution of turbulent energy density in size-space can be expressed by the second-order longitudinal velocity structure function $D_{\parallel}(r)=\left\langle\left[\boldsymbol{v}_{\parallel}\left(\boldsymbol{x}+\boldsymbol{r}\right)-\boldsymbol{v}_{\parallel}\left(\boldsymbol{x}\right)\right]^2\right\rangle_{\boldsymbol{x}}$:

$$f\left(r\right)=\frac{3}{2}\frac{\partial}{\partial r}\left\langle\left[\boldsymbol{v}_{\parallel}\left(\boldsymbol{x}+\boldsymbol{r}\right)-\boldsymbol{v}_{\parallel}\left(\boldsymbol{x}\right)\right]^2\right\rangle_{\boldsymbol{x}}=\frac{3}{2}\frac{\partial}{\partial r}C_2\varepsilon_0^{\frac{2}{3}}r^{\frac{2}{3}}=C_2\,\varepsilon_0^{\frac{2}{3}}r^{-\frac{1}{3}},$$
$$\int_0^{\infty}f\left(r\right)dr=3\left\langle\boldsymbol{v}_{\parallel}{}^2\right\rangle=\left\langle\boldsymbol{v}^2\right\rangle,\qquad \boldsymbol{v}_{\parallel}=\frac{\boldsymbol{r}}{r^2}\boldsymbol{r}\cdot\boldsymbol{v};\quad \left\langle\boldsymbol{v}^2\right\rangle=3\left\langle\boldsymbol{v}_{\parallel}^2\right\rangle;\quad j\left(r\right)=-\varepsilon_0<0. \tag{7}$$

This distribution corresponds to the spectrum proposed by Kolmogorov for stationary homogeneous turbulence. Remarkably, the solution (6) to the fragmentation equation agrees with the Kolmogorov spectrum (7) even including the universal constant ($C_2=2$ was established by measurements). In previous work we have applied symmetry methods in a group-theoretical description of turbulence [3], but on the basis of the Navier-Stokes equation, rather than in the framework of empirico-mathematical model, as in the present paper

References

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