CONTINUOUS SPECTRA AND ENTRAINMENT OF FREE-STREAM VORTICAL DISTURBANCES IN THE ASYMPTOTIC SUCTION BOUNDARY LAYER

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Abstract
Characterizing free-stream disturbances and their entrainment into a shear layer is of a crucial first step towards understanding and predicting receptivity and bypass transition. In some recent studies of bypass transition, continuous modes of the O-S equation have been used to represent free-stream vortical disturbances and the signature induced by them in the boundary layer. For the Blasius boundary layer, a recent study by the present authors shows that continuous modes and entrained disturbances are fundamentally different. The former exhibit some nonphysical features such as 'entanglement of Fourier components' and 'abnormal anisotropy', which are found to be caused by neglecting the non-parallelism. In the present paper, we consider the asymptotic suction boundary layer, which is an exactly parallel flow. Both temporal and spatial continuous spectra may be defined mathematically. However, at a finite R neither of them represents the physical process of free-stream vortical disturbances penetrating into the boundary layer. The latter must instead be characterized by a peculiar type of continuous modes whose eigenfunctions increase exponentially with the distance from the wall. In the limit $R \gg 1$, all three types of continuous spectra are identical at leading order, and hence can be used to represent free-stream vortical disturbances and their entrainment. Low-frequency disturbances are found to generate a large-amplitude streamwise velocity in the boundary layer, which is reminiscent of longitudinal streaks.

BOUNDARY-VALUE PROBLEMS FOR CONTINUOUS SPECTRA AND ENTRAINMENT

The asymptotic suction boundary layer forms over an infinite flat porous plate, through which a steady uniform suction V^* is imposed. The non-dimensional velocity field has the exact solution $(U, V) = (1 - \mathrm{e}^{-y}, -1/R)$, and the Reynolds number $R = U_\infty \delta^*/\nu^* = U_\infty/V^*$; here y is normalized by $\delta^* = \nu^*/V^*$, the displacement thickness of the boundary layer. The flow is perturbed by small-amplitude disturbances

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}) = (u, v, w, p)\, \mathrm{e}^{\mathrm{i}(k_1 x + k_3 z - \omega t)} + c.c.,$$

where $c.c.$ stands for the complex conjugate. The function v satisfies the Orr-Sommerfeld equation

$$\Big\{(D^2 - \bar{k}^2)^2 - \mathrm{i}k_1 R\Big[(U - \omega/k_1)(D^2 - \bar{k}^2) - U''\Big] + (D^2 - \bar{k}^2)D\Big\}v = 0, \tag{1}$$

and the normal vorticity $\Omega = \mathrm{i}k_3 u - \mathrm{i}k_1 w$ is governed by the Squire equation

$$\Big\{\mathrm{i}k_1(U - \omega/k_1) - (D^2 - \bar{k}^2)/R - D/R\Big\}\Omega = -\mathrm{i}k_3 U' v, \tag{2}$$

subject to the boundary conditions $v = \Omega = 0$ at $y = 0$, where $D = \partial_y$. The far-field condition may be written as

$$\left.\begin{array}{ll} v(y) \to & \mathrm{e}^{-\mathrm{i}k_2 y} + B\,\mathrm{e}^{(-1+\mathrm{i}k_2)y} + C\,\mathrm{e}^{-\bar{k}y} \\ \Omega(y) \to & E_1\,\mathrm{e}^{-\mathrm{i}k_2 y} + F_1\,\mathrm{e}^{(-1+\mathrm{i}k_2)y} + k_3 R/(2k_2)\,\mathrm{e}^{-(1+\mathrm{i}k_2)y} \end{array}\right\} \text{ as } y \to \infty, \tag{3}$$

where k_2 is arbitrary, and it will be specified properly when continuous spectra and entrainment are considered.
For a continuous spectrum, the solution of v should remain bounded as $y \to \infty$, and thus k_2 must be real. For a **temporal continuous mode**, k_1 and k_3 are real, and $\omega = \omega_r + \mathrm{i}\omega_i$ is complex. The dispersion relation gives

$$\omega = k_1 + k_2/R - \mathrm{i}(k_1^2 + k_2^2 + k_3^2)/R, \quad c = \omega/k_1 = 1 + k_2/(\omega R) - \mathrm{i}(\omega^2 + k_2^2 + k_3^2)/(\omega R). \tag{4}$$

For a **spatial continuous mode**, ω and k_3 are real, but $k_1 = k_{1,r} + \mathrm{i}k_{1,i}$ is complex. From the dispersion relation, we find

$$k = (\omega - k_2/R)\left[\sqrt{b^2 + \frac{4}{R^2}(\omega - k_2/R)^2} + b\right]^{-1/2} + \mathrm{i}\frac{R}{2}\left\{\left[\sqrt{b^2 + \frac{4}{R^2}(\omega - k_2/R)^2} + b\right]^{1/2} - 1\right\}, \tag{5}$$

with $b = \frac{1}{2}[1 + 4(k_2^2 + k_3^2)/R^2]$.
In the case of **entrainment**, specification of the appropriate far-field condition requires considering vortical disturbances in the free stream, which are convected by the uniform background flow $(U, -1/R)$. Introduce the coordinate system (X, Y), where X and Y are the axes parallel and normal to streamlines in the free stream, respectively. A vortical

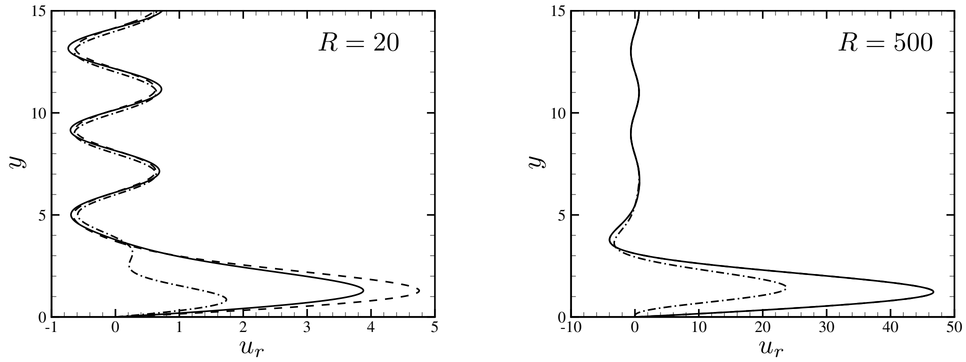

Figure 1. Comparison of the distribution of an entrained disturbance (solid lines) with the eigenfunctions of conventional spatial/temporal continuous modes (dashed/dash-dotted lines) for $\omega = 0.01$ or $k_1 = 0.01$ (temporal mode), $k_2 = \pi/2$ or $K_2 = \pi/2$ (entrained disturbance), $k_3 = \pi/2$ and $E_1 = 1$.

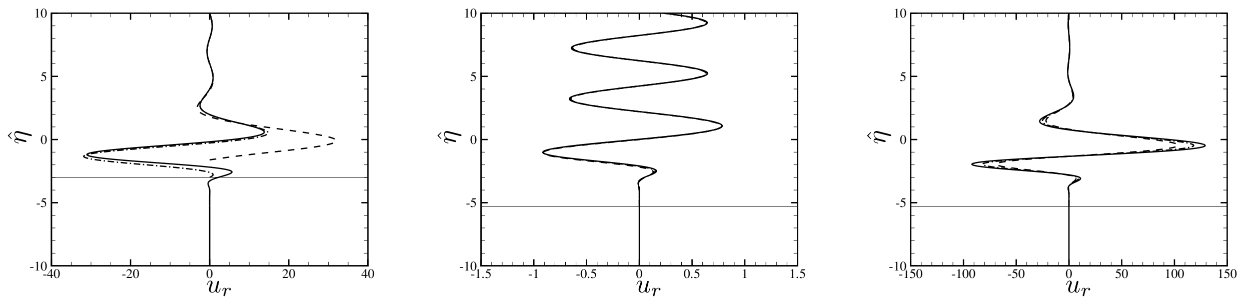

Figure 2. Comparison of the asymptotic and numerical solutions for $\omega = 0.1,\ k_2 = k_3 = \pi/2$ (left), $\omega = 0.4,\ k_2 = k_3 = \pi/2$ (central) and $\omega = 0.4,\ k_2 = -\pi/2,\ k_3 = \pi/2$ at different R. Solid lines: the asymptotic solutions; dashed lines: $R = 500$; dash-dotted lines: $R = 2000$. The thin horizontal lines mark the position of the wall (corresponding to $\hat{\eta} = -\ln(\omega R)$) for $R = 2000$.

disturbance is proportional to $\mathrm{e}^{\mathrm{i}(K_1 X - K_2 Y + k_3 z - \omega t)} + c.c.$, where ω, K_2 and k_3 are real. Using the relation between (x, y) and (X, Y) shows that the far-field condition is given by (3) but with

$$k_2 = K_2 \cos\theta + \omega\Big[\sqrt{b_1^2 + 4\omega^2/R^2} + b_1\Big]^{-1/2} \sin\theta + \mathrm{i}\frac{R}{2}\left\{\Big[\sqrt{b_1^2 + 4\omega^2/R^2} + b_1\Big] - \sqrt{1 + 1/R^2}\right\} \sin\theta, \qquad (6)$$

where $\theta = \tan^{-1}(1/R)$. Now since k_2 is a complex number with $\Im(k_2) > 0$, the amplitude of the component $\mathrm{e}^{-\mathrm{i}k_2 y}$ in (3) increases exponentially in the wall-normal direction. This is a significant difference from the conventional continuous spectra. Thus neither a temporal nor a spatial continuous mode represents the entrainment of a free-stream vortical disturbance at a finite R. In the limit $R \gg 1$, the three boundary-value problems are equivalent to leading order.

A SAMPLE OF NUMERICAL RESULTS

The boundary-value problems consisting of the O-S and Squire equations, (1) and (2) with the far-field conditions (3), are solved numerically. Fig.1 compares the eigenfunctions of spatial/temporal continuous modes with the distribution of an entrained vortical disturbance; for brevity, only the real part of the u-component is shown. For a very low R, e.g. $R = 20$, the three are noticeably different. At a high R, i.e. $R = 500$, the entrained disturbance and the spatial mode are indistinguishable, but the temporal mode is merely qualitatively similar and an appreciable quantitative difference exists. An important feature is that for small ω the streamwise velocity in the boundary layer is much larger than the magnitude of free-stream vortical disturbances. This is the key reason why longitudinal streaks appear in the viscous region of the asymptotic boundary layer. In the limit $\omega R \gg 1$, the entrained disturbance concentrate in the edge layer, where an asymptotic solution may be constructed. For $\omega = O(1)$, the perturbation in the edge layer is comparable with that in the free stream (Fig.2, central). For $R^{-1} \ll \omega \ll 1$, the streamwise velocity acquires a large amplitude (Fig.2, left) so that streaks may appear in the edge layer. We also calculate the response induced by free-stream disturbances with $k_2 < 0$ (Fig.3, right). Interestingly, the response turns out to be much stronger than for $k_2 > 0$.