ON THE APPLICABILITY OF FALKNER–SKAN BOUNDARY LAYER EQUATIONS TO TURBULENT THERMAL CONVECTION

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Abstract To approximate main flow characteristics like the velocity or temperature within the boundary layers which appear in turbulent thermal convection at moderate Rayleigh numbers (up to 10^{14}), one can use the Falkner–Skan boundary layer equations, which are obtained through a generalization of the Prandtl–Blasius ansatz to a non-zero-pressure-gradient case. Within this approach we derive several theoretical estimates with respect to turbulent Rayleigh–Bénard convection and validate them against numerical results obtained in our Direct Numerical Simulations for the operating fluids water and air. In particular it is found that the Falkner–Skan approximation leads to more accurate estimates of the ratio of the thermal and viscous boundary layer thicknesses, compared to the Prandtl–Blasius approach.

MOTIVATION

Turbulent thermal convection between two horizontal plates with lower heated and upper cooled flat surfaces, has been the subject of numerous experimental and numerical studies. This problem is known as turbulent Rayleigh–Bénard convection (RBC) and for recent reviews on it we refer to [1, 5, 2]. The fast development of parallel computer systems allows to investigate already nowadays turbulent RBC in accurate Direct Numerical Simulations (DNS) for the cases, where the core flow is fully turbulent and the boundary layers are laminar or transitional.

To conduct accurate DNS [8, 4, 9] one needs to use fine enough computational meshes, the grid diameter of which is smaller than the Kolmogorov and Batchelor scales. In [7] theoretical estimates of the mesh resolution requirements were derived, which are based on the knowledge of the ratio of the thicknesses of the thermal and viscous boundary layers. The laminar boundary layers were approximated in [7] using the Prandtl–Blasius ansatz, i.e. under the assumption that the wind of turbulence (or Large Scale Circulation – LSC) above the viscous boundary layer is horizontal and constant, which leads to a zero pressure derivative with respect to the wind direction. In contrast to this, DNS [6, 9] of turbulent RBC in air showed that, first, the wind is non-constant along its path and, second, the ratio of the thicknesses of the thermal and viscous boundary layers, although being almost constant along the wind, is approximately two times larger than that predicted by the Prandtl–Blasius equations. In Fig. 1 one can obviously see a LSC in water as well as in air for the Rayleigh number $\mathcal{R}a = 10^8$. Therefore in the present work, in order to account for the influence of the non-constant wind, we make use of the Falkner–Skan approximation of the boundary layers in turbulent thermal convection, which can be interpreted as an extension of the Prandtl–Blasius ansatz to a non-zero pressure change along the wind.

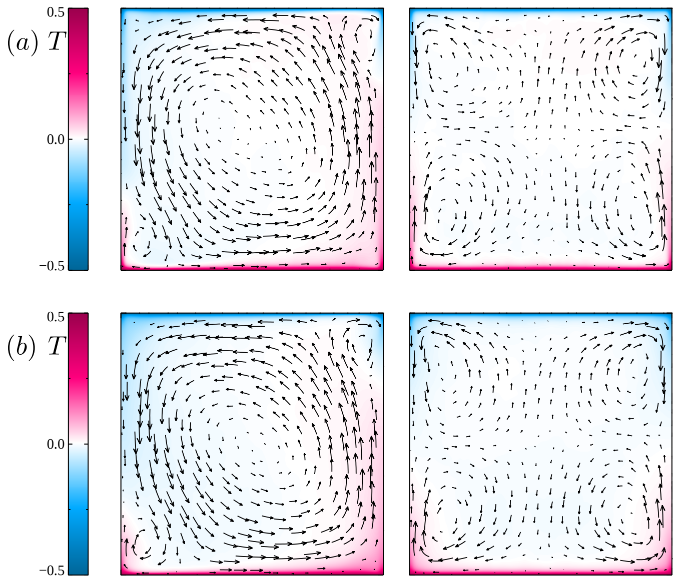

Figure 1. Mean temperature distribution in the plane of the large-scale circulation (left) and orthogonal to it (right) in turbulent Rayleigh–Bénard convection in a cylinder with aspect ratio diameter/height=1 filled with water ($\mathcal{P}r = 4.38$, (a)) and air ($\mathcal{P}r = 0.786$, (b)), as obtained in DNS for $\mathcal{R}a = 10^8$.

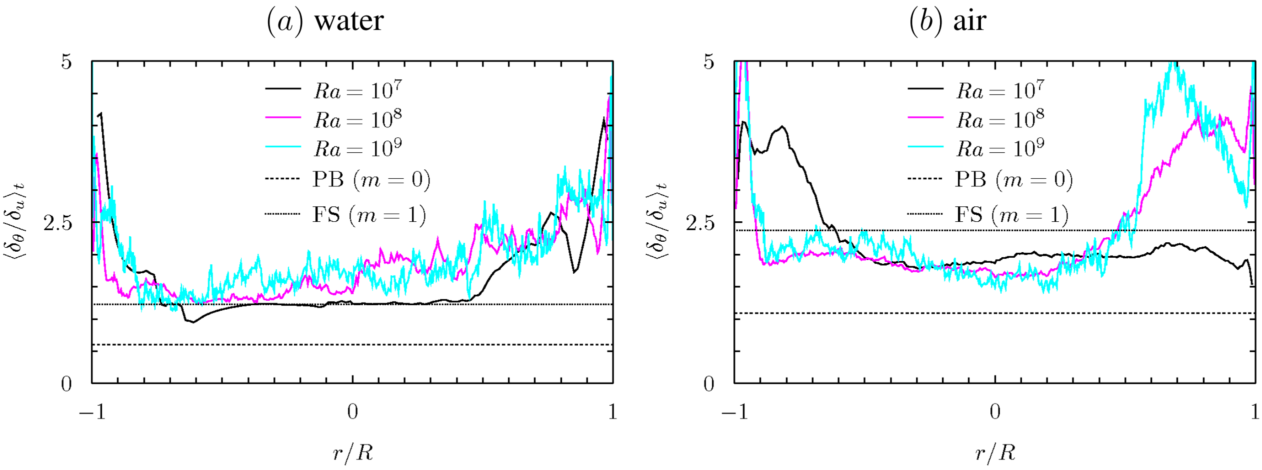

Figure 2. Ratio δ_θ/δ_u of the thermal and viscous boundary layer thicknesses for (a) water and (b) air, as obtained in DNS for different $\mathcal{R}a$ together with the theoretical estimates for $m = 0$ (Prandtl–Blasius flow) and $m = 1$ (stagnation-point flow).

RESULTS

Assuming that the wind above the viscous boundary layer in turbulent RBC can be approximated well with a polynomial function of the horizontal coordinate, one can obtain the following system of boundary layer equations for momentum (Falkner–Skan)

$$\Psi^{'''} + \Psi\Psi^{''} + \frac{2m}{m+1}\left(1 - (\Psi^{'})^2\right) = 0, \quad \Psi(0) = 0, \quad \Psi^{'}(0) = 0, \quad \Psi^{'}(\infty) = 1, \tag{1}$$

and energy

$$\Theta^{''} + \mathcal{P}r \cdot \Psi\Theta^{'} = 0, \quad \Theta(0) = 0, \quad \Theta(\infty) = 1, \tag{2}$$

with respect to a certain similarity variable ξ, which is proportional to the vertical coordinate. Here Ψ and Θ are, respectively, the dimensionless streamfunction and temperature.

One can solve these equations numerically [3, 10]. The solution of the momentum equation (1) depends only on m, while that of the energy one (2) depends also on the Prandtl number $\mathcal{P}r$. The parameter m determines the angle $\phi = \pi/(m+1)$, at which the wind falls at the horizontal heated/cooled plate. Thus $m = 0$ and $m = 1$ correspond, respectively, to a Prandtl–Blasius flow over a horizontal plate and a stagnation-point flow in a right-angle corner. For different m and $\mathcal{P}r$ one obtains different thicknesses of the thermal boundary layers, normalized by the thickness of the viscous one.

According to the above theory, with growing m the ratio of the thicknesses of the thermal and viscous boundary layers also increases. Since in turbulent RBC the wind is not constant and hits the horizontal plate at the angle $\phi < \pi$, the Falkner–Skan approximation with $m \in [0.5; 1]$ generally leads to better predictions of the ratio of the thicknesses, compared to the Prandtl–Blasius approximation. To demonstrate this, in Fig. 2 the DNS results for different Rayleigh numbers for fluids water and air are plotted against the theoretical prediction for $m = 0$ and $m = 1$.

At the conference further results will be presented including the limits of the temperature profiles obtained with the Falkner–Skan ansatz and the grid resolution requirements in DNS of turbulent thermal convection with non-zero pressure gradient.

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