COMPREHENSIVE REALIZABILITY OF PRESSURE STRAIN CORRELATION MODELS

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Abstract The realizability condition acts as a turbulence modeling guideline by attempting to ensure physically meningful predictions of the Reynolds stress tensor. However, these measures to ensure realizability focus solely on the realizability of the Reynolds stresses and do not address the underlying cause of unrealizable predictions, that lies in the unphysical behavior of the unclosed statistical moments in the Reynolds Stress Transport equation. In this investigation, we apply a set of Comprehensive Realizability conditions on the models for the M_{ijkl} tensor, based on popular models, such as those by Launder, Reece and Rodi [4]; Speziale, Sarkar and Gatski [8] and Johansson and Hallback[3]. The regions on the Anisotropy Invariant Map where comprehensive realizability is fulfilled are mapped out for these models, analytically. Thence, we discuss the reasons for the unrealizable regions of all the models. To conclude, the schism between predictive fidelity and realizability is addressed.

OVERVIEW

Introduced by Schumann [7] and Lumley [5], the realizability condition acts as a turbulence modeling guideline by attempting to ensure physically meningful predictions of the Reynolds stress tensor. In this vein, two conceptions of this condition exist, expressly, the Weak and the Strong realizability conditions. Both of these address the dynamics of the Reynolds stresses in the vicinity of the 2-Component limit of the Anisotropy Invariant Map.
It has been proved, mathematically [2] and computationally [6], that linear models cannot ensure physically meningful solutions in all regions of the Anisotropy Invariant Map. This has been one of the principal motivations to append non-linear terms to the model expression. The pressure strain correlation model of Johansson and Hallback [3] invokes a fifth order expression with respect to the Reynolds stress anisotropies and ensures, in the classical sense, realizability everywhere.
However, these measures to ensure realizability focus solely on the realizability of the Reynolds stresses. Consequently, they do not address the underlying cause of unrealizable predictions, that lies in the unphysical behavior of the unclosed statistical moments in the Reynolds Stress Transport equation. [1] has avered that to ensure realizability, the model for each unclosed statistical moment be individually realizable. In this regard, a set of "comprehensive realizability" conditions have been obligated upon the M_{ijkl} tensor, scilicet:

1. $M_{\alpha\alpha kl}$=0;
2. $\frac{\partial U_j}{\partial x_i}\frac{\partial U_k}{\partial x_l}M_{iljk} \geq 0$;
3. $\left|\frac{\partial U_k}{\partial x_l}M_{\alpha l\alpha k}\right| \leq \sqrt{2\langle u_\alpha u_\alpha\rangle}\sqrt{\frac{\partial U_j}{\partial x_i}\frac{\partial U_k}{\partial x_l}M_{iljk}}$.

The first condition is just a restatement of the Strong realizability condition at the 2-C limit. The second condition ensures the positive semi-definite nature of M_{ijkl}. The third condition forces the M tensor to be large enough so as to ensure the Schwarz inequality applied to the RPSC. Thus, unlike the classical realizability conditions, these are applicable for all regions of the Anisotropy Invariant Map.
It is expedient to model this forth order tensor, as just ensuring the adherence of a RPSC model to said guidelines does not lead to a physically consistent expression for the M tensor. Analogous to the RPSC, certain constraints are obligated upon such a representation. These include:

1. linearity condition, with respect to the Reynolds stress anisotropies;
2. symmetry conditions, with respect to the indices ($M_{ijkl} = M_{jikl}, M_{ijkl} = M_{ijlk}$);
3. continuity condition ($M_{ijil} = 0$);
4. Green's condition ($M_{ijkk} = \frac{R_{ij}}{2k}$);
5. Crow constraint.

A model for the M tensor that ensures adherence to all these conditions is classified as "Self-Consistent". The most general ansatz for a linear model of M, which has the imprimatur of the classical framework of pressure strain correlation modeling, is:

$$M_{ijpq} = A_1\delta_{ij}\delta_{pq} + A_2(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) + A_3\delta_{ij}b_{pq} + A_4\delta_{pq}b_{ij} + A_5(\delta_{ip}b_{jq} + \delta_{iq}b_{jp} + \delta_{jp}b_{iq} + \delta_{jq}b_{ip}). \quad (1)$$

We apply the Comprehensive Realizability conditions, sequentially, to the M tensors based on popular models, such as [4], [8] and [3]. It is analytically proved that linear models can ensure the positive semi-definite nature of the M tensor for weak to moderate levels of anisotropy, only.

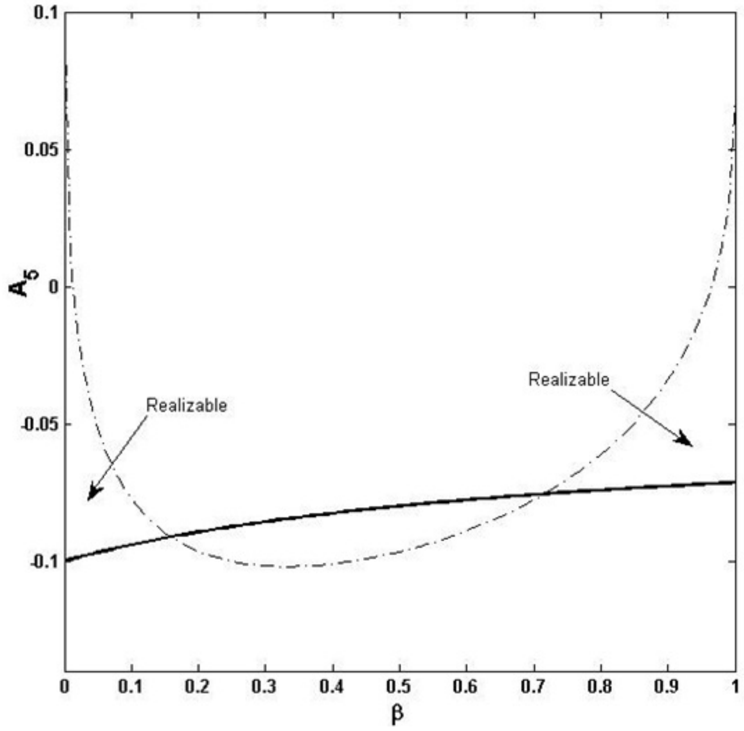

Figure 1. Realizable ranges of the A_5 coefficient for linear models, vis-à-vis the psd nature of the M tensor.

The regions on the Anisotropy Invariant Map where comprehensive realizability is fulfilled are mapped out for all varities of models. Thence, we discuss the reasons for the unrealizable regions of all the models. To conclude, the schism between predictive fidelity and realizability is discussed.

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