THE COMPLEX UNSTEADY FLOW WITHIN A FLUID FILLED ANNULUS AND ITS TRANSITION TO TURBULENCE

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Abstract The analysis of turbulence in transient flows has applications across a broad range of fields. The flow of fluid in a toroidal container is a paradigm for studying the complex dynamics due to the turbulence in these transient flows. We consider both the 'spin-up' problem, in which the toroidal container is spun up from rest to a constant angular frequency, and the 'spin-down problem', in which the system is already in a state of rigid-body rotation and is spun down linearly to rest. These two approaches allow us to examine the development of an impulsively generated axisymmetric boundary-layer, adjacent to the interior annular wall, its subsequent instability and the larger scale transient features within this class of flows.

FORMULATION

We consider an annulus of square cross-section with a centreline radius L and cross-sectional radii a and b, in the $\hat{r}$- and $\hat{z}$-directions respectively, filled with an incompressible viscous fluid of kinematic viscosity ν, as shown in Figure 1. In the 'spin-up' problem, the fluid and annulus are initially at rest, that is, the initial angular frequency, Ω_i, is zero. At time $t = 0$, an unsteady flow is generated by an impulsive change in the rotation rate of the annulus to an angular frequency Ω_f.

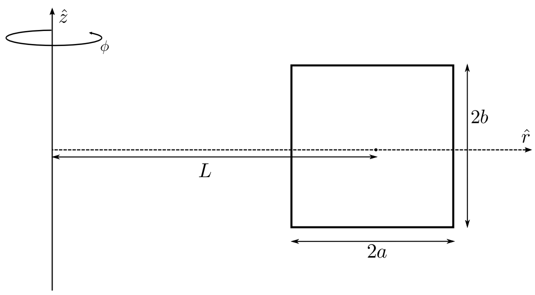


Figure 1: A cross-section of the annulus, with dimensional centreline radius L and cross-sectional radii a and b, in the r- and z-directions respectively. The outer wall of the annulus is at $\hat{r} = L + a$, whilst the inner wall is at $\hat{r} = L - a$. The upper wall of the annulus is at $\hat{z} = b$, with the lower wall is at $\hat{z} = -b$.

For the initial boundary-layer analysis, it is most convenient to work in a cylindrical coordinates system, in which we have chosen $2a$ as the typical length scale, $2a\Omega_f$ as the typical velocity scale, Ω_f^{-1} as the time scale, with pressure non-dimensionalised on the inertial scale, $4a^2\Omega_f^2\rho$. The dimensionless velocity components corresponding to the coordinates (r, ϕ, z) are labelled (u, w, v) and, for the spin-up problem, the system is initially in a state of rest:

$$u \equiv v \equiv w \equiv 0 \quad \text{at } t = 0. \tag{1}$$

The boundary conditions for $t > 0$ are that the annulus rotates at the new frequency:

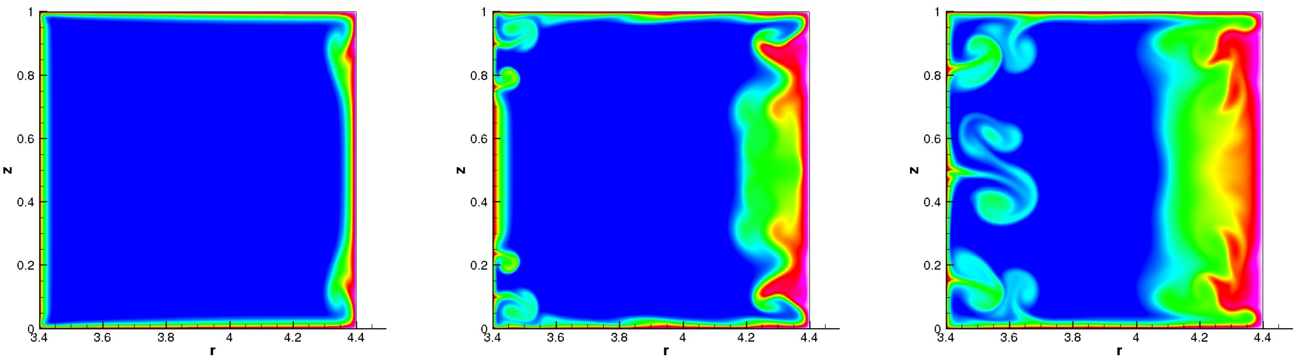
$$u = v = 0, \quad w = \left(\frac{2\left(\Omega_f - \Omega_i\right) t}{\sqrt{\left(t_s - t\right)^2 + t_s + t}} + \Omega_i \right) r \tag{2}$$
$$\text{on } r = \tfrac{1}{2}\left(-1 + \delta^{-1}\right),\ \tfrac{1}{2}\left(1 + \delta^{-1}\right) \text{ and } z = -\tfrac{\alpha}{2},\ \tfrac{\alpha}{2},$$

where t_s is the time taken for the annulus to spin-up to its final angular frequency Ω_f, $\delta = a/L$ is a curvature parameter, and $\alpha = b/a$ is the aspect ratio of the annular cross-section.

This investigation is based on the experimental work of Madden and Mullin [1], and the subsequent work by Hewitt *et al.* [3], which used a toroidal pipe of radius $a = 16$ mm and centreline radius of curvature $L = 125$mm, which gives a curvature parameter, $\delta = a/L = 0.128$. We shall use this value in all curvature-dependent computations presented.

ROTATIONALLY SYMMETRIC NAVIER-STOKES COMPUTATIONS

The rotationally symmetric Navier-Stokes equations are solved numerically. We exploit the numerical capabilities of *Semtex* [2], a quadrilateral spectral element DNS code, ideal for solving problems in cylindrical coordinates systems.



(a) $t = 2$ (b) $t = 4$ (c) $t = 6$

Figure 2: Contour plots of the azimuthal velocity component w in the case of a toroidal container spun up from rest. Show are three time snaps taken at (a) $t = 2$, (b) $t = 4$ and (c) $t = 6$ for a flow with a Reynolds number $Re = 2000$.

The resulting complex dynamics can be seen in the contour plots presented in figure 2. At time $t = 2$, the time at which the container has reached its final (constant) angular frequency, we observe the initial boundary layer development (Stewartson layers on the inner and outer wall, and Ekman layers on the upper and lower boundaries), together with the signature of a boundary-layer eruption (an unsteady separation) within the inner Stewartson layer. At a later time, $t = 4$, the inner Stewartson layer has developed an instability akin to a Görtler instability (as described by Otto [4]). This vortex instability then serves to shed vorticity into the Ekman layers at the upper and lower boundaries which manifests as a nonlinear travelling wave within the Ekman layers. These travelling waves subsequently collide with the Stewartson layer on the inner boundary; at this time in our simulations the Stewartson layer has itself broken down, exhibiting the type of boundary-layer separation described by Hewitt *et al.* [3]. At even later times, $t = 6$ in figure 2, the flow has developed a secondary asymmetric instability at the inner wall, an instability that has all the hall marks of the secondary instability that develops upon fully nonlinear Görtler vortices [5].

We will describe the boundary-layer development, including the initial ejection and subsequent instability, as well as travelling waves in the Ekman layers which arise at the upper and lower boundaries of the container and the vortex shedding which occurs in the Stewartson layers for higher Reynolds numbers. We will also elucidate the 'oscillatory front' that moves across the cross-section ultimately leading to rigid-body rotation. The differences between the case of spin-up and spin-down will be highlighted.

References

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