LOCALIZED UNSTABLE MODES IN A PRECESSING SPHERE

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Abstract The asymptotic analysis of the linear stability of the steady flow in a precessing sphere is performed in the double limit of rapid rotation and weak precession. It is shown that the disturbance localized in the critical region gives the stability boundary which agrees with the laboratory experiment.

A precessing sphere We consider the flow of an incompressible viscous fluid in a sphere which is spinning with constant angular velocity Ω_s and precessing with another constant angular velocity Ω_p perpendicular to the spin. The flow characteristics of this system is determined by two nondimensional parameters, the Reynolds number $Re = a^2\Omega_s/\nu$ and the Poincaré number $\Gamma = \Omega_p/\Omega_s$, where a is the sphere radius and ν is the kinematic viscosity of fluid. Although this flow has long attracted peoples's attention as a simple model of rotating celestial bodies especially with relation to the geophysical applications [1] and the compact turbulence generator [2], the fundamental properties, such as the structure of the steady flows, their instability boundaries, the state distribution, etc. over the whole parameter range have not been studied systematically yet. Here we investigate the stability characteristics of the steady flow of this system.

Stability of steady flows It is well-known [3] that the flow in a sphere spinning with a constant angular velocity is neutrally stable in the inviscid limit and that all the disturbances damp with decay rate proportional to $Re^{-1/2}$ for $Re \gg 1$. Since the Coriolis force is proportional to Γ, it is natural to expect that the flow would be destabilized if the pressession is as strong as $\Gamma = O(Re^{-1/2})$. As a matter of fact, the stability analysis [4] of the **global** disturbances (the eigenfunctions of which spread over the whole sphere) give us the neutral curve, $\Gamma = 7.9Re^{-1/2}$ (see the broken line in figure 1). On the other hand, the stability analysis by DNS (now in progress by the present author) as well as by laboratory experiment [5] indicate a steeper curve, $\Gamma \propto Re^{-\alpha}$) with $\alpha \approx 0.8$ (see the solid line and many symbols along it in figure 1). The **purpose** of the present work is to derive this power law theoretically.

Flow structure Before going into the stability analysis it would be approriate to give a brief summary of the flow structure in the limit $\Gamma \ll Re^{-1/2} \ll 1$ (see figure 2). Since the direction of the torque by the Coriolis force is different from the spin axis and $Re \gg 1$, a thin boundary layer of thickness δ ($\equiv Re^{-1/2}$) develops over the whole sphere surface except near the critical circle (where the boundary-layer approximation is broken down) located at 60^o apart from the spin axis [6, 7]. There appears another tiny region called the **critical region** of thickness of $O(\delta^{4/5})$ and width of $O(\delta^{2/5})$ [8]. The tangential and the radial components of velocity in this region are of $O(\Gamma\delta)$ and $O(\Gamma\delta^{3/5})$ respectively. Furthermore, two conical shear layers of thickness of $O(\delta^{2/5})$ develop from the two critical regions which focus at the poles of the spin axis. The velocity in these layers are of $O(\Gamma\delta^{3/5})$ and parallel to the layers. The velocity profile (which will be used in the stability analysis below) has been obtained very recently [9]. With reference to this flow structure we may expect three different kinds of **local** disturbance modes sitting either in the critical region, in the conical shear layers, or in the boundary layer. In the following we consider the disturbance localized in the critical region.

Eigenvalue problem of localized modes The governing equations for the small disturbance (put in the form $\boldsymbol{u}\mathrm{e}^{\sigma t}$ and $P\mathrm{e}^{\sigma t}$) are written, in the precession frame, as

$$\sigma\boldsymbol{u} + (\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{u}} + \nabla P + 2\Gamma\widehat{\boldsymbol{z}}\times\boldsymbol{u} - \delta^2\nabla^2\boldsymbol{u} = 0, \qquad \nabla\cdot\boldsymbol{u} = 0 \tag{1}$$

with the boundary condition $\boldsymbol{u} = 0$ on $r = 1$, where $\overline{\boldsymbol{u}} = r\sin\theta\widehat{\boldsymbol{\phi}} + \gamma\delta\overline{u}_\xi^{(1)}\widehat{\boldsymbol{r}} + \gamma\delta^{3/5}\overline{u}_\eta^{(1)}\widehat{\boldsymbol{\theta}} + \gamma\delta^{3/5}\overline{u}_\phi^{(1)}\widehat{\boldsymbol{\phi}}$ is the steady flow, the explicit expressions of $\overline{u}_\xi^{(1)}$, $\overline{u}_\eta^{(1)}$ and $\overline{u}_\phi^{(1)}$ are given in [9]. Here, the disturbence has been assumed to be local in the critical region, and the scaled coordinates (ξ,η) has been introduced through $r = 1 - \delta^{4/5}\xi$ and $\cos\theta = \frac{1}{2} + \delta^{2/5}\eta$, and (r,θ,ϕ) is the spherical polar coordinate with the x-axis being the polar axis. Now we put the Poincaré number as $\Gamma = \gamma\delta^{8/5}$ (we have checked that all the other choice of the power dependence would not lead to meaninful result!). We expand the disturbance fields and the growth rate σ in power series of $\delta^{1/5}$ as $u_r = \delta^{2/5}\widetilde{u}_\xi + \delta^{3/5}\widetilde{u}_\xi^{(1)} + \delta^{4/5}\widetilde{u}_\xi^{(2)} + \cdots$, $u_\theta = \widetilde{u}_\eta + \delta^{1/5}\widetilde{u}_\eta^{(1)} + \delta^{2/5}\widetilde{u}_\eta^{(2)} + \cdots$, $u_\phi = \widetilde{u}_\phi + \delta^{1/5}\widetilde{u}_\phi^{(1)} + \delta^{2/5}\widetilde{u}_\phi^{(2)} + \cdots$, $P = \delta^{2/5}\widetilde{P} + \delta^{3/5}\widetilde{P}^{(1)} + \delta^{4/5}\widetilde{P}^{(2)} + \cdots$, $\sigma = \widetilde{\sigma} + \delta^{1/5}\sigma^{(1)} + \delta^{2/5}\sigma^{(2)} + \cdots$. These expansions are substituted in the governing equations (1). The leading order in the expansion can be solved easily. It turns out that $\widetilde{\sigma}^{(1)}$ is pure imaginary so that the stability of the disturbance depends on the sign of the real part of $\widetilde{\sigma}^{(2)}$, which is determined by the following eigenvalue problem derived by the use of the solvability condition in the second order:

$$\left(\mathcal{L}_0 + \gamma^2\mathcal{L}_1 - \widetilde{\sigma}^{(2)}\frac{\partial^2}{\partial\xi^2}\right)\Psi_+ + \gamma^2\mathcal{L}_2\Psi_- = 0, \qquad \left(\mathcal{L}_0^* + \gamma^2\mathcal{L}_1^* - \widetilde{\sigma}^{(2)}\frac{\partial^2}{\partial\xi^2}\right)\Psi_- + \gamma^2\mathcal{L}_2^*\Psi_+ = 0 \tag{2}$$

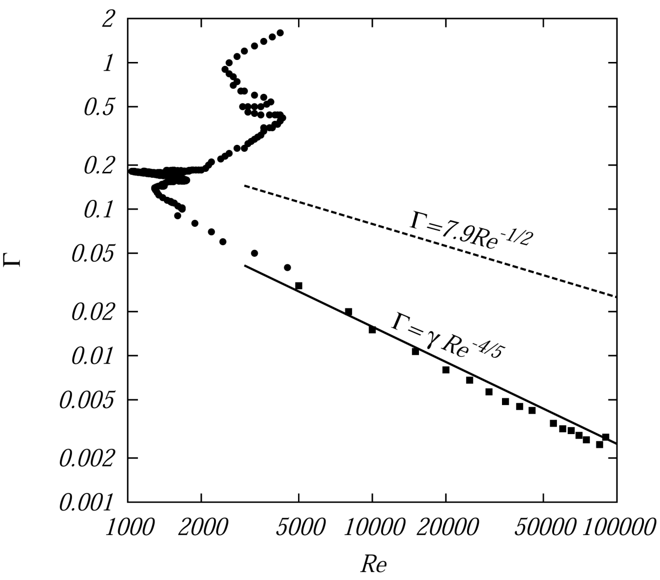

Figure 1. Stability boundary of steady flow in a precessing sphere. DNS (circles), Experiment (squares) Theory with local (solid line) and global modes (broken line). (Experimental data were taken from [5]). The coefficient γ is to be determined by solving (1) – (4).

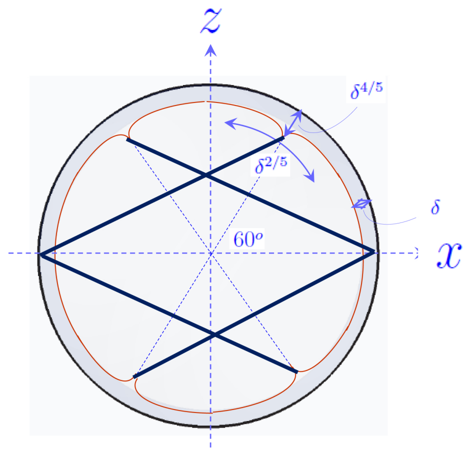

Figure 2. Flow structure. x is the spin axis, and z is the precession axis. $\delta = Re^{-1/2}$.

with boundary condition,

$$\Psi_\pm(0,\eta) = \frac{\partial \Psi_\pm}{\partial \xi}(0,\eta) = 0, \quad \Psi_\pm(\infty,\eta) = 0, \quad \frac{\partial^3 \Psi_\pm}{\partial \xi^3}(0,\eta) \pm \frac{3\mathrm{i}}{2}\frac{\partial \Psi_\pm}{\partial \eta}(\infty,\eta) = 0. \tag{3}$$

Here

$$\left.\begin{aligned} &\mathcal{L}_0 = \frac{\partial^4}{\partial \xi^4} + 2\mathrm{i}\eta\frac{\partial^2}{\partial \xi^2} - \frac{3\mathrm{i}}{2}\frac{\partial^2}{\partial \xi \partial \eta}, \quad \mathcal{L}_1 = A\frac{\partial^3}{\partial \xi^3} + B\frac{\partial^3}{\partial \xi^2 \partial \eta} + C\frac{\partial^2}{\partial \xi^2} + D\frac{\partial^2}{\partial \xi \partial \eta} + E\frac{\partial}{\partial \xi} + F\frac{\partial}{\partial \eta}, \\ &\mathcal{L}_2 = G\frac{\partial^2}{\partial \xi^2} + H\frac{\partial^2}{\partial \xi \partial \eta} + I\frac{\partial}{\partial \xi} + J\frac{\partial}{\partial \eta}, \end{aligned}\right\} \tag{4}$$

and $A(\xi,\eta) \sim J(\xi,\eta)$ are given functions derived from the steady velocity field [9]. The critical value γ for which the real part of $\widetilde{\sigma}^{(2)}$ vanishes will be obtained by solving equations (1) – (4). The calculation is under way, and the results will be presented at the conference.

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