NEW CONSERVATION LAWS FOR HELICALLY SYMMETRIC FLOWS AND THEIR IMPORTANCE FOR TURBULENCE THEORY

Olga Kelbin[1], Alexei Cheviakov [2], Martin Oberlack [1,3] & Ivan Delbende [4]
[1] *Chair of Fluid Dynamics, TU Darmstadt, Petersenstr. 30, 64287 Darmstadt, Germany*
[2] *Dep. of Mathematics and Statistics, University of Saskatchewan, Saskatoon, S7N 5E6, Canada*
[3] *GS Computational Engineering, TU Darmstadt, Dolivostr. 15, 64293 Darmstadt, Germany*
[4] *Université Pierre et Marie Curie-Paris 6, LIMSI-CNRS, UPR 3251, BP 133, 91403 Orsay Cedex, France*

Abstract Our present understanding of statistical $3D$ turbulence dynamics for reasonable large wave numbers (or small scales) largely relies on the dissipation of kinetic energy a quantity which is invariant under all symmetry groups of Navier-Stokes equations except some special combination of the scaling groups. On the other hand in $2D$ turbulence, which is translational invariant in one direction, the transfer mechanism among scales is rather different since the vortex stretching is non-existing. Instead, the transfer and scale determining key invariant is enstrophy: an area integral of the vorticity squared which is one of the infinite many integral invariants of $2D$ inviscid fluid mechanics (see 4 below). Hence the basic mechanism between and $2D$ and $3D$ turbulence is very different. To close this gap we consider flows with a helical symmetry which is a twist of translational and rotational symmetry. The resulting equations are "$2\frac{1}{2}D$" which means they have three independent velocity components though only two independent spatial variables. We presently show that the helically symmetric equations of motion admit an infinite number of new non-trivial conservation laws (CL). It is to be expected that the new CLs may give some deeper insight into turbulence dynamics and hence bridging $2D$ and $3D$ turbulence.

CONSERVATION LAWS

Starting from the usual cylindric coordinate system in the variables (r, z, φ) we define the helical coordinate ξ as follows:

$$\xi = az + b\varphi \tag{1}$$

(see Figure 1), where the (r, ξ) is considered a reduced helically symmetric system. From (1) we observe two limiting cases, namely the cases of rotationally symmetric ($a = 1, b = 0$) and plane ($a = 0, b = 1$) flows. Re-writing the Euler and Navier-Stokes equations as well as the related vorticity transport equations in the helical reduced setting we derived local conservation using the direct construction method [1]. This comprises various new sets of CLs for both inviscid and viscous flows, including families that involve arbitrary functions. In case of all three velocities being non-zero we can find an infinite set of vorticity-related CLs for inviscid and viscous cases. In particular, for inviscid flows, we obtain a family of conserved quantities that generalize helicity with its usual definition $h = \boldsymbol{u} \cdot \boldsymbol{\omega}$:

$$\begin{aligned} \frac{\partial}{\partial t}\left(h\, H\left(\frac{r}{B}u^\eta\right)\right) + \nabla \cdot \Big[H\left(\frac{r}{B}u^\eta\right)\left[\boldsymbol{u} \times \nabla E + (\boldsymbol{\omega} \times \boldsymbol{u}) \times \boldsymbol{u}\right] \\ + E u^\eta \boldsymbol{e}_{\perp\eta} \times \nabla H\left(\frac{r}{B}u^\eta\right)\Big] = 0. \end{aligned} \tag{2}$$

Here $H\left(r/Bu^\eta\right)$ is an arbitrary function of the velocity u^η along the helix ξ. As the CL (2) is not of a material type (see below) we have distinct dynamics for every choice of H (if $H = 1$, this CL reduces to the conservation of helicity). We expect that the key integral invariant for the "helical $2\frac{1}{2}D$ turbulence" will be one of the quantities of this family of CLs.

Secondly, we also consider the special case of two-component flows, with zero velocity component in the invariant direction ($u^\eta = 0$) (see Figure 2). Per definition the vorticity has only one component in this case, namely ω^η. For such type of flow we can find an extended set of conserved quantities. In particular, the well-known infinite set of generalized enstrophy CLs, which hold for plane flows, also holds for the general two-component helically invariant flows:

$$\frac{\partial}{\partial t}\left(T\left(\frac{B}{r}\omega^\eta\right)\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r u^r T\left(\frac{B}{r}\omega^\eta\right)\right) + \frac{1}{B}\frac{\partial}{\partial \xi}\left(u^\xi T\left(\frac{B}{r}\omega^\eta\right)\right) = 0, \tag{3}$$

where $T(\cdot)$ is a time independent arbitrary function. The infinite-dimensional family of CLs (3) corresponds to a family of Casimirs and is of material type as any value for $T\left(\frac{B}{r}\omega^\eta\right)$ assigned to a particle at $t = 0$ is conserved for $t > 0$; the case $T(q) = q^2$ may be referred to as enstrophy conservation in two-component helical flows. Formulas (3) generalize the well-known plane two-component flow CLs

$$\frac{\partial}{\partial t}\left(N(\omega^z)\right) + \frac{\partial}{\partial x}\left(u^x N(\omega^z)\right) + \frac{\partial}{\partial y}\left(u^y N(\omega^z)\right) = 0 \tag{4}$$

for $a = 0, b = 1$ and, moreover, gives rise to a infinite family of CLs for axisymmetric flows $a = 1, b = 0$ which is given

$$\frac{\partial}{\partial t}\left(S\left(\frac{1}{r}\omega^{\varphi}\right)\right) + \frac{\partial}{\partial r}\left(u^r S\left(\frac{1}{r}\omega^{\varphi}\right)\right) + \frac{\partial}{\partial z}\left(u^z S\left(\frac{1}{r}\omega^{\varphi}\right)\right) = 0, \tag{5}$$

where $N(\cdot)$ and $S(\cdot)$ are time independent arbitrary functions.

At this point large scale computations of the helically symmetric Navier-Stokes equations are under way in order to validate some of the theoretical findings using a new helical DNS code [2]. In particular, it is intended to see, for the general helically symmetric flow, how some of the conserved quantities for Euler equations from the family (2) behave in the viscous setting in the limit $\nu \to 0$. Secondly, it is also explored if for the reduced case ($u^{\eta} = 0$) some of the quantities $T(\cdot)$ in (3) are analogous to the enstrophy ω_z^2 for plane flows, which belongs to family (4) and is known to be a global invariant of $2D$ inviscid flows.

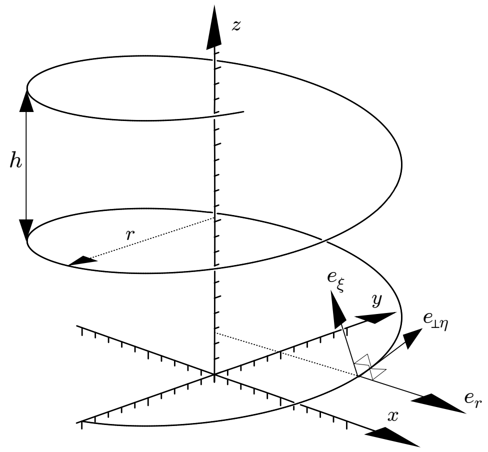

Figure 1. Helical coordinates with their respective unit vectors in r-, ξ and η-directions.

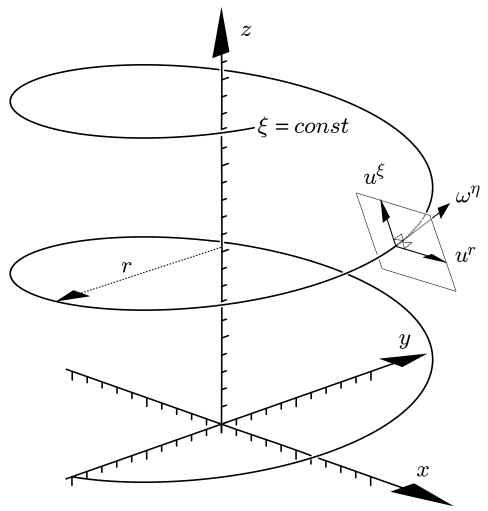

Figure 2. Orientation of the helical velocity components and the vorticity in 2D case.

References

[1] S.C. Anco, G.W. Bluman, and A.F. Cheviakov. *Applications of Symmetry Methods to Partial Differential Equations*, **168**. Springer: Applied Mathematical Sciences, 2010.

[2] I. Delbende, M. Rossi, and O. Daube. Dns of flows with helical symmetry. *Theor. Comput. Fluid Dyn.*, **26**:141–160, 2012.