UNIVERSAL FRICTION LAW FOR TURBULENT BOUNDARY LAYERS WITH WALL SUCTION

Igor Vigdorovich
Institute of Mechanics, Moscow State University, Moscow, Russia

Abstract A universal friction law for turbulent boundary layers on a flat plate with suction is established. Experimental skin-friction distributions obtained for various suction factors and Reynolds numbers are described in similarity variables by a single universal curve. The law is valid for the entire range of suction velocities from zero one till the limiting values corresponding to asymptotic-suction boundary layers. The analysis is not based on any particular turbulence model.

PROBLEM FORMULATION

We consider an incompressible turbulent boundary-layer flow over a smooth flat plate with a constant free-stream velocity u_e and a constant wall suction velocity v_w parallel to the normal vector. The origin of the Cartesian coordinate system is situated at the leading edge of the plate.

Closure condition. In the flow under consideration, any mean quantity is a function of Cartesian coordinates x and y and only three governing parameters: the kinematic viscosity ν, the free-stream velocity u_e, and the suction velocity v_w. This means that the gradient of the stream-wise mean-velocity component, the turbulent shearing stress, and the boundary-layer thickness can be represented in terms of universal functions:

$$\frac{\partial u}{\partial y} = F_1(x,\, y,\, \nu, u_e, v_w), \quad \langle u'v'\rangle = F_2(x,\, y,\, \nu, u_e, v_w), \quad \Delta = F_3(x,\, \nu, u_e, v_w). \tag{1}$$

We solve the first and third equations in (1) for x and u_e and substitute them into the second one to obtain the relation $\langle u'v'\rangle = F_4(\Delta,\, y,\, \nu,\, \partial u/\partial y,\, v_w)$, which after applying dimensional considerations yields

$$\langle u'v'\rangle = -\left(y\frac{\partial u}{\partial y}\right)^2 S(\mathrm{Re},\, \beta,\, \eta), \quad \mathrm{Re} = \frac{y^2}{\nu}\frac{\partial u}{\partial y}, \quad \beta = \frac{v_w}{y\partial u/\partial y}, \quad \eta = \frac{y}{\Delta}. \tag{2}$$

Here, S is some universal function of three variables. The local Reynolds number Re is defined here as the ratio of characteristic turbulent and molecular viscosity values, the parameter β characterizes suction affect on shearing stress.

Change of variables. The mean-flow stream function $\psi(x,\, y)$ satisfies the boundary-layer equation for zero-pressure-gradient flow:

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = (\nu\psi_{yy} - \langle u'v'\rangle)_y\,; \quad x>0, \quad y=0: \quad \psi_y = 0, \quad \psi_x = -v_w; \quad y\to\infty: \quad \psi_y \to u_e. \tag{3}$$

We perform in this equation the following change of variables [1]: $\psi = u_e\Delta\Psi(\xi,\, \eta)$, $\xi = \ln \mathrm{Re}_\Delta$, where $\mathrm{Re}_\Delta = u_e\Delta/\nu$, using the logarithm of the Reynolds number Re_Δ based on the boundary-layer thickness and the normalized distance from the wall η as independent variables. With the closure condition (2), Eq. (3) becomes

$$\Lambda[\Psi_\eta\Psi_{\xi\eta} - (\Psi+\Psi_\xi)\Psi_{\eta\eta}] = \left[(\eta\Psi_{\eta\eta})^2\, S(\mathrm{Re},\, \beta,\, \eta) + e^{-\xi}\Psi_{\eta\eta}\right]_\eta, \quad \mathrm{Re} = e^\xi\eta^2\Psi_{\eta\eta}, \quad \beta = B\,(\eta\Psi_{\eta\eta})^{-1};$$

$$\xi > -\infty, \quad \eta = 0: \quad \Psi = 0, \quad \Psi_\eta = 0, \quad \Lambda(\Psi+\Psi_\xi) = -B; \quad \eta\to\infty: \quad \Psi_\eta \to 1. \tag{4}$$

Here, $B = v_w/u_e$ is the suction factor. Besides the normalized stream function Ψ, Eq. (4) contains the second unknown function $\Lambda(\xi) = \mathrm{d}\Delta/\mathrm{d}x$. We will seek an asymptotic solution to the problem (4) as $\xi\to\infty$, i. e. for high values of the logarithm of the Reynolds number based on the boundary-layer thickness. We introduce a small parameter ε and a new independent variable $\zeta = \varepsilon\xi$, $1/\zeta = O(1)$. We specify the suction factor as $B = \varepsilon^2 b$, $b = O(1)$.

SOME RESULTS

Wall region. In the outer part of the wall region, in the logarithmic sublayer, the velocity profile obeys the similarity law

$$\frac{2}{v_+}\left(\sqrt{1+v_+u_+} - 1\right) = \frac{1}{\varkappa}[\ln y_+ + C(v_+)] + O\left(y_+^{-\alpha}\right), \quad y_+\to\infty, \quad \alpha > 0. \tag{5}$$

Here, the accustomed wall variables are used, in particular $v_+ = B/\sqrt{\frac{1}{2}c_f}$, where c_f is the skin-friction coefficient, and $\varkappa = \sqrt{S(\infty,\, 0,\, 0)}$. For $v_+ = 0$, in the limiting case of an impermeable wall, Eq. (5) reduces to the normal log law, hence $\varkappa$ is the von Kármán constant equal to 0.41 and $C(v_+) = C_0 + O(v_+)$, $v_+\to 0$, where $C_0 = 2.05$. Since $B = O(\varepsilon^2)$ and

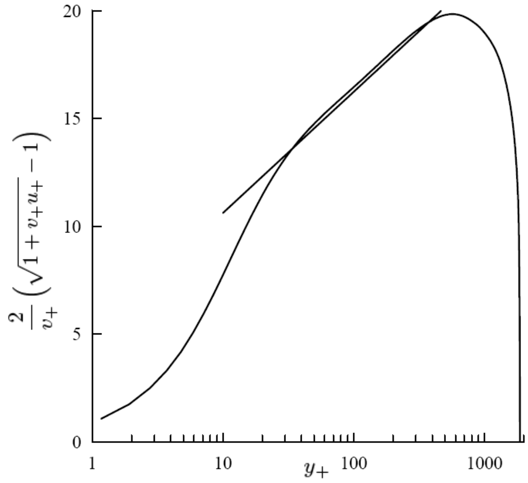

Figure 1. Velocity profile near the suction wall in turbulent Poiseuille flow with a constant wall transpiration [2] plotted in similarity variables (5). The straight line has the equation $\frac{1}{0.41}(\ln y_+ + 2.05)$.

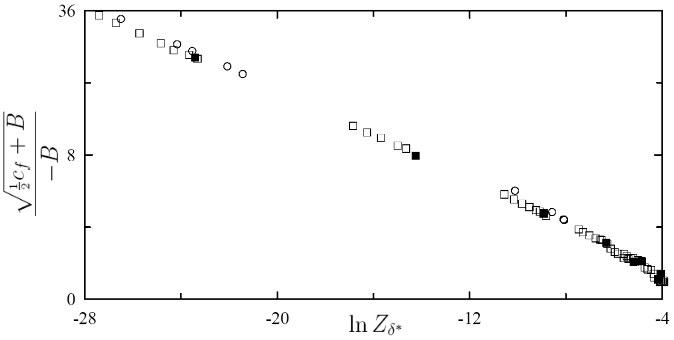

Figure 2. Experimental data on skin friction [3, 4] plotted in terms of scaling variables (8); ○, [3], □, [4] (indirect measurements), ■, [4] (measured with floating elements).

$\sqrt{\frac{1}{2}c_f} = O(\varepsilon)$, the parameter $v_+ = O(\varepsilon)$, i. e. is always a small quantity. The similarity law (5) is perfectly supported by the recently obtained DNS data [2]. Figure 1 shows the calculated velocity profile for the pressure-driven turbulent plane channel flow (Poiseuille flow) with a constant transverse mass flux. The velocity profile near the suction wall corresponds to $v_+ = -0.0305$. Plotted in variables (5), it has a distinct logarithmic potion, which is very close to that for the velocity profile near an impermeable wall.

Outer region of the boundary layer. In the outer region of the boundary layer we seek the solution in the form of following asymptotic expansions:

$$\Lambda(\xi) = -\varepsilon^2 b\,\lambda(s) + O(\varepsilon^3), \quad \Psi(\xi,\,\eta) = \Psi_w(\xi) + \eta - \varepsilon^2 b\,g(s,\,\eta) + O(\varepsilon^3), \quad s = \frac{1}{\varepsilon}\left(\zeta - \frac{2\varkappa}{\sqrt{-b}}\right) - k\ln(-b\varepsilon^2), \quad (6)$$

where k is a certain coefficient. After substituting Eqs. (6) in Eq. (4) we obtain for the functions g and λ a partial differential equation

$$\Big[(\eta g_{\eta\eta})^2\, S(\infty,\,\beta,\,\eta)\Big]_\eta + (1+\eta\lambda)g_{\eta\eta} = \lambda g_{s\eta}, \quad \beta = -(\eta g_{\eta\eta})^{-1}; \quad g(s,\,0) = g_\eta(s,\,\infty) = g_{\eta\eta}(s,\,\infty) = 0. \quad (7)$$

Friction law. The universal friction law is obtained as a result of asymptotic matching of the solutions for the outer and wall regions making use of the inner-solution asymptotics (5) and the corresponding wall asymptotics of the solution to the boundary-value problem (7). The law has the following formulation:

$$Z_{\left\{\substack{\delta^*\\ x}\right\}} \equiv (-B)^{\mp 1}\sqrt{\frac{c_f}{2}}\exp\left(\frac{2\varkappa\sqrt{\frac{1}{2}c_f}}{B}\right)\mathrm{Re}_{\left\{\substack{\delta^*\\ x}\right\}} = \Phi_{\left\{\substack{1\\ 2}\right\}}\left(\frac{\sqrt{\frac{1}{2}c_f}+B}{-B}\right) + O(\sqrt{c_f}). \quad (8)$$

Here, Re_x and Re_{δ^*} are the Reynolds numbers based on the distance from the leading edge of the plate and the boundary-layer displacement thickness, respectively, Φ_1 and Φ_2 are some universal functions. It is valid for the entire range of suction velocities from zero one till the limiting values corresponding to asymptotic-suction boundary layers and, as figure 2 shows, is in perfect agreement with experimental data [3, 4]. The data cover a vide range of Reynolds numbers $4 \leqslant \mathrm{Re}_x \cdot 10^{-5} \leqslant 20$ and suction factors $1.2 \leqslant -B\cdot 10^3 \leqslant 2.4$ for [3] and $3.8 \leqslant \mathrm{Re}_x \cdot 10^{-5} \leqslant 35$ and $1 \leqslant -B\cdot 10^3 \leqslant 3.6$ for [4]. Plotted in terms of similarity variables (7), all experimental points follow a single curve in figure 2.

References

[1] I. I. Vigdorovich. Asymptotic analysis of turbulent boundary layer flow on a flat plate at high Reynolds numbers. *Fluid Dyn.* **28**: 514–523, 1993.

[2] V. S. Avsarkisov, M. Oberlack, G. Khujadze, I. Vigdorovich. Lie symmetry based scaling laws and DNS verification of a turbulent Poiseuille flow with uniform wall transpiration. 9th Euromech Fluid Mech. Conf. Book of Abstracts, Rome 2012.

[3] R. L. Simpson, R. J. Moffat, W. M. Kays. The turbulent boundary layer on a porous plate: experimental skin friction with variable injection and suction. *Intern. J. Heat and Mass Trans.* **12**(7): 771–789, 1969.

[4] K. Depooter. *The measurement of wall shear stress on a porous plate with mass transfer using a floating element technique and the investigation of various indirect measuring methods.* PhD Thesis, Canada, Univ. Waterloo, 1973.