NONLOCAL PRESSURE CONTRIBUTIONS TO THE SMALL-SCALE STATISTICS OF HOMOGENEOUS ISOTROPIC TURBULENCE

Michael Wilczek[1] & Charles Meneveau[1]
[1]*Department of Mechanical Engineering & Institute for Data Intensive Engineering and Science (IDIES), Johns Hopkins University, Baltimore, Maryland 21218*

Abstract The topology and statistics of the small scales of homogeneous isotropic turbulence can comprehensively be described in terms of the velocity gradient tensor. Here, one of the key problems is to quantify the nonlocal pressure contributions, which enter the velocity gradient tensor dynamics in form of the pressure Hessian. This nonlocality, which tightly interacts with the local self-amplification mechanisms, also poses severe challenges on the closure of statistical models for the small scales of turbulence. In this paper, we systematically elaborate the statistical structure of the pressure Hessian and explicitly evaluate it for Gaussian random fields. The results then are compared to those obtained by direct numerical simulations, and possible implications for improved closure schemes are discussed.

Studying the statistical properties of the velocity gradient tensor $A_{ij}(\boldsymbol{x},t) = \frac{\partial u_i}{\partial x_j}(\boldsymbol{x},t)$ provides a rich characterization of the small scales of turbulent flows. The evolution of this tensor is given by

$$\frac{\partial}{\partial t}\mathrm{A}(\boldsymbol{x},t) + \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\mathrm{A}(\boldsymbol{x},t) = -\mathrm{A}(\boldsymbol{x},t)^2 - H(\boldsymbol{x},t) + \nu\Delta\mathrm{A}(\boldsymbol{x},t)\,, \tag{1}$$

which states that the velocity gradient is advected by the velocity field and additionally is subject to self-amplification, nonlocal pressure Hessian effects ($H_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}p$, where p denotes the kinematic pressure) and viscous dissipation. Similar to the Burgers equation, the self-amplification term, considered on its own, can lead to a steepening of the velocity gradient and eventually to a finite-time singularity. According to DNS observations (see, e.g., [4]), the viscous dissipation and pressure Hessian, however, counteract this unbounded growth and significantly interact with the local self-amplification. While the viscous term mainly has a damping effect, the pressure Hessian participates in the dynamics in a highly non-trivial manner. For stationary flows an additional large-scale forcing can be considered, which we omit here and in the following.

The statistical properties of the velocity gradient tensor can comprehensively be described in terms of the corresponding probability density function (PDF), for which exact evolution equations are readily derived [5, 8, 7]. In this framework, the evolution of the probability density function can be tracked by the statistical evolution of the velocity gradient given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} = -\mathcal{A}^2 - \big\langle \mathrm{H}\big|\mathcal{A}\big\rangle + \big\langle \nu\Delta\mathrm{A}\big|\mathcal{A}\big\rangle\,. \tag{2}$$

The appearing conditional averages $\langle \mathrm{H}|\mathcal{A}\rangle$ and $\langle \nu\Delta\mathrm{A}|\mathcal{A}\rangle$ represent the mean values of the pressure Hessian and the viscous diffusion, respectively, when averaging a class of fluid particles with fixed value of $\mathcal{A}$. Once these terms are expressed as functions of $\mathcal{A}$, closure of the statistical equations is achieved. The evolution equation (2) is closely related to a broad class of statistical models as summarized in [6], and shares particular similarity to the one proposed by Chevillard and Meneveau [3]. The main difference is found in the fact that this evolution equation describes the evolution of a *class* of fluid particles rather than the evolution of an *individual* particle. In the statistical sense, these two approaches are equivalent.

Investigation of the PDF equations also reveals that the pressure Hessian term can be closed with the help of two-point information. This can be seen by expressing the pressure Hessian by the solution of the Poisson equation, which after evaluating the conditional average leads to

$$\big\langle H_{ij}\big|\mathcal{A}\big\rangle = \frac{2}{3}\mathcal{Q}\,\delta_{ij} + \frac{1}{2\pi}\int_{\mathrm{PV}}\mathrm{d}\boldsymbol{r}\left[\frac{\delta_{ij}}{r^3} - 3\frac{r_i r_j}{r^5}\right]\big\langle Q(\boldsymbol{x}',t)\big|\mathcal{A}\big\rangle\,, \tag{3}$$

where $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$ denotes the distance vector and $Q = -\frac{1}{2}\mathrm{Tr}(A^2)$ denotes one of the invariants of the velocity gradient tensor. The first term on the right-hand side is the local contribution to the pressure Hessian, which is also taken into account in the Restricted Euler model [1]. The remaining integral constitutes the nonlocal pressure Hessian contribution. Probably the most interesting observation about this relation is that, in order to close this tensorial expression, only the scalar function $\langle Q(\boldsymbol{x}',t)|\mathcal{A}\rangle$, which depends on the distance vector $\boldsymbol{r}$ and the velocity gradient tensor at position $\boldsymbol{x}$ only, has to be specified. As a consequence, no matter which model assumption is made for the conditional Q, the structure of the integration kernel will ensure a number of desired features, such as symmetry with respect to a interchange of indices $(i \leftrightarrow j)$ and a vanishing trace of the nonlocal contribution. Furthermore, with this result closures can analytically be evaluated assuming that turbulent fields belong to a certain class of random fields.

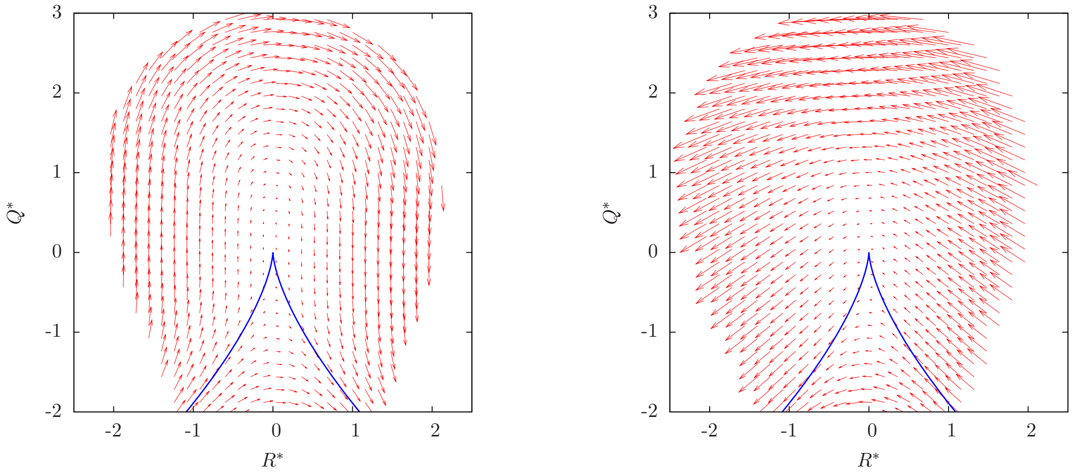

Figure 1. Left: restricted Euler contribution in the $\mathcal{R}$-$\mathcal{Q}$ plane, and right: projection of the nonlocal Gaussian pressure Hessian contribution to the $\mathcal{R}$-$\mathcal{Q}$ plane. (See also [2] for a comparison to a numerically obtained Gaussian random field.) The blue line represents the Viellefosse line.

In particular, we can evaluate the nonlocal pressure Hessian contributions for Gaussian random fields, which yields the remarkably simple result

$$\langle H_{ij}|\mathcal{A}\rangle = -\frac{26}{105}\,\mathcal{Q}\,\delta_{ij} - \frac{8}{105}\,\left(\mathrm{Tr}[\mathcal{S}^2 - \mathcal{W}^2]\right)\,\delta_{ij} - \frac{8}{5}\,\mathcal{W}^2_{ij} - \frac{8}{7}\,\mathcal{S}^2_{ij}\,, \tag{4}$$

where $\mathcal{S} = \frac{1}{2}\left(\mathcal{A} + \mathcal{A}^T\right)$ is the rate-of-strain tensor and $\mathcal{W} = \frac{1}{2}\left(\mathcal{A} - \mathcal{A}^T\right)$ the rate-of-rotation tensor. That means, a non-trivial closed expression for the pressure Hessian is found in terms of the velocity gradient tensor and its (anti-)symmetric contributions. While turbulence certainly is non-Gaussian, evaluating this Gaussian approximation is one of the few systematic ways of gaining insights into plausible ansatzes for the nonlocal pressure contributions, which is an interesting approach in its own right. For example, one interesting feature about this closure is that when calculating the dynamics of the invariants from the ODE system (2) with the closure (4), additional invariants apart from $\mathcal{R}$ and $\mathcal{Q}$ will be involved. Still, the effect of this nonlocal contribution is probably best understood by considering its projection to the $\mathcal{R}$-$\mathcal{Q}$-plane, which is shown in Fig. 1 along with the local Restricted Euler contribution. While the Restricted Euler contribution potentially induces a singularity along the right tail of the Viellefosse line, this effect is counter-balanced by the Gaussian nonlocal pressure Hessian contribution. The Gaussian closure, however, may lead to unphysical behavior in $\mathcal{R} < 0$ half plane. To fully understand this, however, the full dynamical system (2) has to be considered, which is work in progress. In this paper, we will discuss the properties of the Gaussian closure in detail. Furthermore, taking this approximation as a starting point, we will also explore non-Gaussian generalizations and discuss the properties of the resulting closure.

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