NONLOCAL MODEL OF SUPERFLUID TURBULENCE

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Abstract In this paper, we build up a nonlocal model of superfluid turbulence. The theory chooses as fundamental fields the density, the velocity, the energy density, the heat flux, and the averaged vortex line length per unit volume. The constitutive quantities are assumed to depend on the fundamental fields and on their first derivative, allowing to describe mechanical and thermal dissipations and vortex diffusion. The restrictions on the constitutive relations are deduced from the entropy principle, using the Liu method of Lagrange multipliers.

INTRODUCTION

In recent years there has been growing interest in superfluid turbulence, because a better understanding of it can throw new light on problems in classical turbulence, but it is also crucial to explain much observed superfluid behavior, including that relevant for the application of superfluid helium as a coolant for superconducting devices. Quantum turbulence is described as a chaotic tangle of quantized vortices of equal circulation κ; κ is called quantum of vorticity and results $\kappa = h/m_4$, with h the Planck constant, and m_4 the mass of ^{4}He atom: $\kappa \simeq 9.97\ 10^{-4}$cm^2/s. The vortex tangle is assumed to be isotropic and may be described by introducing a scalar quantity L, the average vortex line length per unit volume, briefly called vortex line density and whose dimensions are $(length)^{-2}$. A first study of this interesting phenomenon was made in [1]. In that work, the presence of vortices was modelled through a pressure tensor for which a constitutive relation was written. In a successive work [2], a model of inhomogeneous superfluid turbulence was formulated, which does not consider nonlocal effects.

The aim of this paper is the formulation of a more general model of superfluid turbulence that takes into account the non local behaviour. As in [2], we will choose as fundamental fields the density ρ, the velocity $\mathbf{v}$, the energy density $E = \rho\epsilon$, the heat flux $\mathbf{q}$, and the averaged vortex line length per unit volume L. The starting point is a general set of balance equations for the fields $\Gamma = (\rho, \rho\mathbf{v}, \rho\epsilon + \frac{1}{2}\rho v^2, \mathbf{q}, L)$, that in terms of non convective quantities are written as:

$$\begin{cases} \dot\rho + \rho\nabla\cdot\mathbf{v} = 0, \\ \rho\dot{\mathbf{v}} + \nabla\cdot\mathbf{J}^{\mathbf{v}} = 0, \\ \dot E + E\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{q} + \mathbf{J}^{\mathbf{v}}\cdot\nabla\mathbf{v} = 0, \\ \dot{\mathbf{q}} + \mathbf{q}\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{J}^{\mathbf{q}} = \sigma^{\mathbf{q}}, \\ \dot L + L\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{J}^L = \sigma^L, \end{cases} \tag{1}$$

where $\mathbf{J}^{\mathbf{v}}$ the stress tensor, $\mathbf{J}^{\mathbf{q}}$ the flux of the heat flux, and $\mathbf{J}^L$ the flux of vortex lines. In this system, an upper dot denotes the material time derivative. In (3) $\sigma^{\mathbf{q}}$ and σ_L are terms describing the net production of heat flux and vortices, for which the constitutive relations:

$$\sigma_{\mathbf{q}} = -KL\mathbf{q}, \qquad \sigma_L = -BL^2 + AqL^{3/2}, \tag{2}$$

will be chosen in agreement with [2]. To describe nonlocal effects, the constitutive equations for the fluxes are assumed to depend on the fundamental fields and on their first derivative. The spatial inhomogeneities require new terms in the description of the system. As a consequence of the material objectivity principle the expressions of the fluxes are:

$$\begin{aligned} \mathbf{J}^{\mathbf{v}} &= (p + \eta_1\nabla\cdot\mathbf{v} + \eta_2\nabla\cdot\mathbf{q})\mathbf{U} + \lambda_1 <\nabla\mathbf{v}> + \lambda_2 <\nabla\mathbf{q}> \\ \mathbf{J}^{\mathbf{q}} &= (\beta_0 + \beta_1\nabla\cdot\mathbf{v} + \beta_2\nabla\cdot\mathbf{q})\mathbf{U} + \xi_1 <\nabla\mathbf{v}> + \xi_2 <\nabla\mathbf{q}> \\ \mathbf{J}^L &= \nu_0\mathbf{q} + \nu_1\nabla\rho + \nu_2\nabla E + \nu_3\nabla L. \end{aligned} \tag{3}$$

where p, η_h, λ_h, β_h, ξ_h, and ν_h are functions of ρ, E, L, and they introduce new terms in the balance equations (1) with respect to the previous model. Angular brackets denote the deviatoric part of the tensors $\nabla\mathbf{v}$ and $\nabla\mathbf{q}$.

In the work, further restrictions on these constitutive relations are obtained imposing the validity of the second law of thermodynamics, applying the Liu method of Lagrange multipliers [3]. This method requires the existence of a scalar function S and a vector function $\mathbf{J}^S$ of the fundamental fields, namely the entropy density and the entropy flux density respectively, such that the following inequality:

$$\begin{aligned} \dot S &+ S\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{J}^S - \Lambda^\rho\left[\dot\rho + \rho\nabla\cdot\mathbf{v}\right] - \Lambda^{\mathbf{v}}\cdot\left[\rho\dot{\mathbf{v}} + \nabla\cdot\mathbf{J}^{\mathbf{v}}\right] \\ &- \Lambda^E\left[\dot E + E\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{q} + \mathbf{J}^{\mathbf{v}}\cdot\nabla\mathbf{v}\right] - \Lambda^{\mathbf{q}}\cdot\left[\dot{\mathbf{q}} + \mathbf{q}\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{J}^{\mathbf{q}} - \sigma^{\mathbf{q}}\right] \\ &- \Lambda^L\left[\dot L + L\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{J}^L - \sigma^L\right] \geq 0, \end{aligned} \tag{4}$$

is satisfied for arbitrary fields ρ, $\mathbf{v}$, E, $\mathbf{q}$ and L. The quantities Λ^{ρ}, $\Lambda^{\mathbf{v}}$, Λ^{E}, $\Lambda^{\mathbf{q}}$ and Λ^{L} are Lagrange multipliers, which are also objective functions of $\rho, E, \mathbf{q}, L$ and their first derivative.

In the paper the consequences of inequality (4) on the constitutive relations will be exploited and a set of field equations will be written. The equations obtained will be applied, in following papers, to the study of Poiseuille flow of liquid helium II in a cylindrical channel, generalizing the study made in previous papers [4, 5], where the inhomogeneities of line density L were neglected.

The model, with the necessary modifications, could be applied also to the study of turbulence in classical fluids, with very high Reynolds numbers.

References

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