WEAKLY-NONLINEAR INSTABILITY DEVELOPMENT IN A SHARPLY STRATIFIED SHEAR FLOW WITH AN INFLECTION-FREE VELOCITY PROFILE

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Abstract Within the framework of weakly nonlinear approach, we study the development of an ensemble of unstable waves with so close phase velocities that the waves share a common critical layer. It is shown that their amplitudes increase, as a rule, explosively. During the initial stage the three-wave interaction is dominant and the low-frequency part of the spectrum grows faster. Later, when the higher-order interactions come into play, the growth of the high-frequency part of the spectrum is accelerating in such a way that to the end of the weakly nonlinear stage all the waves come with the amplitudes of order unity.

LINEAR STABILITY

In incompressible, stably stratified flows, with velocity $V_x = U(z)$ which increases monotonically (without inflection points) upwards, from zero at the bottom $(z = 0;\ U'(0) = 1)$ to $U_0 = 1$ as $z \to \infty$, and the squared buoyancy frequency $\Omega^2(z) = Jn(z)$ (where $\int_0^\infty \mathrm{d}z\, n(z) = 1$) localized in a pycnocline with the thickness $\ell \ll 1$ centered at $z = z_N = \int_0^\infty \mathrm{d}z\, z n(z) = O(1)$, an inviscid instability takes place resulting in the growth of a wide spectrum of disturbances [1, 2]. When the stratification level (described by the bulk Richardson number J) is rather low, oblique waves (which have a wave vector $\mathbf{k} = (k, q, 0)$ with $|q/k| = O(1)$ or greater) are predominantly amplified by the instability whereas when $J = O(\ell^{3/2})$ a wave with a streamwise $\mathbf{k}$ (i.e., $q = 0$) becomes the most unstable. It should be emphasized that if $\ell^2 \ll J < \ell$, in a wide part of the instability domain the growth rate has only a weak dependence on $\mathbf{k}$, and phase velocities of waves differ from $U_N = U(z_N)$ only by $O(\ell)$ (see Figure 1). Therefore, the fastest-growing waves have a common critical layer (hereinafter CL) with the pycnocline embedded in it.

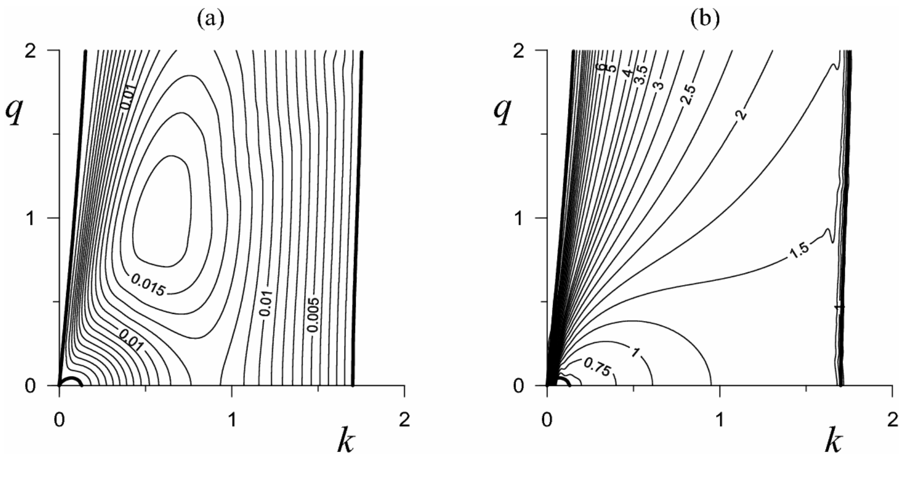

Figure 1. Level lines of (a) growth rate and (b) deviation $(c - U_N)/\ell$ of the phase velocity from U_N. The instability domain boundaries are shown in bold lines; $\ell = 0.04$ (halocline), $J = 0.01$.

A WEAKLY-NONLINEAR STAGE OF THE INSTABILITY DEVELOPMENT

At this stage, a significant contribution to the evolution can be made only by resonance interactions, i.e., those which involve the waves with a common CL (so-called phase-locked modes, see [3], for example). Because in flows under consideration the pycnocline is inside the CL, the result of such interactions is described by nonlinear evolution equations (hereinafter NEEs) which have the form of integral equations [4, 5].

Let us begin with the early stage of a weakly-nonlinear evolution where the main contribution to NEEs is due to three-wave interaction. In this process, interacting waves (with wave vectors obeying the relation $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$ and common CL) grow in such a manner that the most high-frequency wave ($\mathbf{k}_1$) increases with the linear growth rate and parametrically accelerates the growth of the other two. For example, in the case of an isolated triad their growth becomes super-exponential. For this reason, in contrast to the usual case, such an interaction should be interpreted as the development of the waves $\mathbf{k}_2$ and $\mathbf{k}_3$ under catalytic action of the wave $\mathbf{k}_1$ (compare with [3]) rather than as 'decay' $(\mathbf{k_1} \to \mathbf{k_2} + \mathbf{k_3})$ and 'fusion' $(\mathbf{k_2} + \mathbf{k_3} \to \mathbf{k_1})$ of waves forming the triad [5].

To study the evolution of ensembles consisting of more than three waves we impose the (commonly accepted in numerical simulation) condition of the periodicity in x and y, with periods $2\pi / k_0$ and $2\pi / q_0$, respectively. Then the ensemble should have a discrete spectrum, $\omega_m = m\omega_0 = mk_0 U_N$, where $1 \le m \le M$ and the frequency of the catalyzing wave $\Omega = M\omega_0$. If $M \ge 3$, the waves of such an ensemble interact both with the catalyzing wave and between themselves, and a qualitative analysis of NEEs shows that whilst the growth of catalyzing wave remains exponential, amplitudes of other waves should increase explosively, $A_\mathbf{k}(t) \sim (t_* - t)^{-\alpha(\mathbf{k}) + i\beta(\mathbf{k})}$, as well as the CL thickness, $L \sim (t_* - t)^{-1}$. Notice that during this stage the low-frequency waves grow faster because the growth index $\alpha(\mathbf{k}) = \alpha_m = 3(M - m)/(M - 2)$ decreases with m (down to zero as $m = M$). Numerical solutions of NEEs for $M = 3, 5, 6$ demonstrate an explosive growth of amplitudes with these indices as well. It should be noted, however, that $M = 6$ ensembles containing, among others, the waves with frequencies $\omega^{(1)} = \Omega / 3$ and $\omega^{(2)} = 2\Omega / 3$ evolve in two steps. In the beginning, the fast growth of those waves takes place, with indices $\alpha^{(1)} \approx 6$ and $\alpha^{(2)} \approx 3$ typical of $M = 3$ rather than $M = 6$ ensembles, and then these amplified waves interact with other waves and accelerate their growth until the $M = 6$ growth indices are reached. It seems likely that such a behavior is inherent in all ensembles with a sufficiently wide spectrum. In particular, the analysis of the Holmboe wave spectra obtained in numerical simulation and laboratory experiment ([6], Figure 8) demonstrates that during the initial stage of evolution one can clearly see local maxima of intensity at $k \approx k_c / 3$ and $k \approx 2k_c / 3$ where k_c is the streamwise wave number of the fastest-growing wave.

As soon as the amplitude of the lowest-frequency wave attains $O(L^{5/2})$ the higher-order nonlinear interactions come into play, and the trend of perturbation development changes. All the waves, and the catalyzing one among them, are now growing explosively with the growth index $\alpha_m = 2 + m/2$ increasing with the frequency in such a manner that towards the end of the weakly-nonlinear stage all the amplitudes become of order unity.

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