SYSTEMATICS OF TURBULENCE IN THE DISSIPATIONLESS, UNFORCED, 2D, FOURIER-TRUNCATED GROSS-PITAEVSKII EQUATION

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Abstract
We carry out a systematic, direct numerical simulation (DNS) of the two-dimensional, Fourier-truncated, Gross-Pitaevskii equation to study the turbulent evolutions of its solutions for a variety of initial conditions. We find that the time evolution of this system can be classified into four regimes, which have qualitatively different statistical properties. In the first regime there are transients that depend on the initial conditions; in the second, power-law scaling regions, in the energy and the occupation-number spectra, appear and start to develop; the exponents of these power-laws and the extents of the scaling regions change with time and depended on the initial condition; in the third regime, the spectra drop rapidly for modes with wave numbers $k > k_c$, and partial thermalization takes place for modes with $k < k_c$; the self-truncation wave number $k_c(t)$ depends on the initial conditions and it grows either as a power of t or as $\log t$; finally, in the fourth regime, complete thermalization is achieved and correlation functions and spectra are consistent with their nontrivial Berezinskii-Kosterlitz-Thouless forms, if we account for finite-size effect.

INTRODUCTION

The elucidation of the nature of superfluid turbulence has continued to engage the attention of physicists. Experimental realizations of such turbulence have been obtained in the bosonic superfluid ^{4}He, its fermionic counterpart ^{3}He, and Bose-Einstein condensates (BECs) of cold atoms in traps and their optical analogues. Theoretical and numerical studies have used a variety of models for superfluid turbulence; these include the two-fluid model, Biot-Savart-type models, and the Gross-Pitaevskii equation. These models have been studied by a combination of theoretical methods, such as wave-turbulence theory, and numerical simulations. Most of these studies have been restricted to three dimensions (3D); numerical studies of two-dimensional (2D) models for superfluid turbulence have been increasing over the past few years. We carry out a direct numerical simulation (DNS) of the dissipationless, unforced, Fourier-truncated, 2D, GP equation with a view to identifying which features of the turbulent evolution of the solutions of this equation are universal, insofar as they do not depend on initial conditions. Our perspective is different from that of earlier studies of the 2D GP equation; in particular, we elucidate in detail the dynamical evolution of this system and examine the various stages of its thermalization.

MODEL, INITIAL CONDITIONS, AND NUMERICAL METHODS

The GP equation, which describes the dynamical evolution of the wave function ψ of a weakly interacting 2D Bose gas at low temperatures, is

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t} = -\nabla^2\psi(\mathbf{x},t) + g|\psi|^2\psi(\mathbf{x},t); \tag{1}$$

$\psi(\mathbf{x},t)$ is a complex, classical field and g is the effective interaction strength. This equation conserves the total energy $E = \int_{\mathcal{A}}\left[|\nabla\psi|^2 + \frac{1}{2}g|\psi|^4\right]d^2x$ and the total number of particles $N = \int_{\mathcal{A}}|\psi|^2 d^2x$, where $\mathcal{A} = L^2$ is the area of our 2D, periodic, computational domain of side L. To perform a systematic, pseudospectral, direct numerical simulation (DNS) of the spatiotemporal evolution of the 2D, Fourier-truncated, GP equation, we have developed a parallel, MPI code in which we discretize $\psi(\mathbf{x},t)$ on a square simulation domain of side $L = 32$ with N_c^2 collocation points [1]. We use periodic boundary conditions, because we study homogeneous, isotropic turbulence in this 2D system; and employ a fourth-order, Runge-Kutta scheme, with time step Δt, for time marching. The parameters for some of our DNS runs are given in Table 1. To initiate turbulence in the 2D, GP equation we use three types of initial conditions `IC1`, `IC2`, and `IC3`, always normalized to correspond to a total number of particles $N = 1$. The first of these is best represented in Fourier space as follows:

$$\widehat{\psi}(\mathbf{k},t=0) = \frac{1}{\sqrt{\pi^{1/2}\sigma}}\exp\left(-\frac{(k-k_0)^2}{2\sigma^2}\right)\exp\left(i\Theta(k_x,k_y)\right), \tag{2}$$

where $k = \sqrt{k_x^2 + k_y^2}$, $\Theta(k_x,k_y)$ are random numbers distributed uniformly on the interval $[0,2\pi]$; $k_0 = \mathcal{N}_0\Delta k$ and $\sigma = \mathcal{B}\Delta k$, where the integer $\mathcal{N}_0$ controls the spatial scale at which energy is injected into the system, and the real number $\mathcal{B}$ specifies the Fourier-space width of $\widehat{\psi}$ at time $t = 0$. The initial condition `IC2` is like `IC1` but, in addition, it has a finite

	N_c	$k_0(\times\Delta k)$	$\sigma(\times\Delta k)$	g	N_0^i	E
A1	1024	5	2	1000	–	2.120
B1	128	5	1	10000	0.95	5.44
B2	128	5	1	1000	0.95	0.59

Table 1. Parameters for three of our DNS runs `A1` (`IC1`), `B1` (`IC2`) and `B2` (`IC2`).

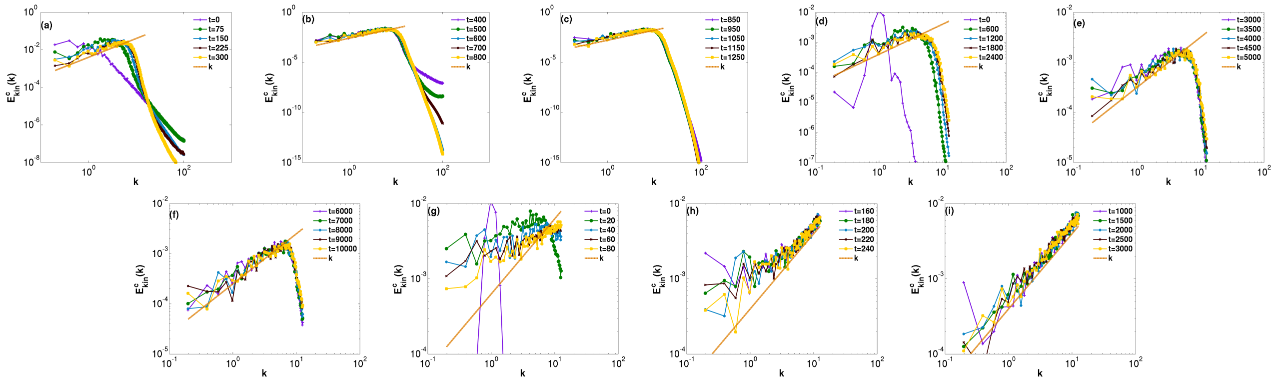

Figure 1. Log-log (base 10) plots of the compressible-kinetic-energy spectra $E^c_{kin}(k)$ from our DNS runs (a)-(c) `A1`, (d)-(f) `B2`, and (g)-(i) `B1` at different times t (indicated by curves of different colours); a power law $\sim k$ is shown by orange-dashed lines.

initial condensate population $N_0^i = \mid \widehat{\psi}(\mathbf{k} = 0, t) \mid^2 (\Delta k)^2$ at time $t = 0$. We obtain the initial condition `IC3` by solving the 2D, stochastic, Ginzburg-Landau equation [1].

RESULTS

We find that the dynamical evolution of the dissipationless, unforced, 2D, Fourier-truncated GP equation can be classified, roughly, into the following four regimes, which have qualitatively different statistical properties: (1) The first is the region of initial transients; this depends on the initial conditions. (2) This is followed by the second regime, in which we see the onset of thermalization; here the energy and occupation-number spectra begin to show power-law-scaling behaviours, but the power-law exponent and the extents of the scaling regions change with time and depend on the initial conditions. (3) In the third regime, which we call the region of partial thermalization, these spectra show clear, power-law, scaling behaviours, with a power that is independent of the initial conditions, and, at large wave vectors, an initial-condition-dependent, self-truncation regime, where spectra drop rapidly; (4) finally, in the fourth regime, the system thermalizes completely and exhibits correlation functions that are consistent with the predictions of the Berezinskii-Kosterlitz-Thouless (BKT) theory, if the simulation domain and simulation time are large enough. Although some of these regimes have been seen in some earlier numerical studies of the 2D GP equation, we are not aware of any study that has systematized the study of these four dynamical regimes. In particular, regime 3, which shows partial thermalization and self-truncation in spectra, has not been identified in the 2D, Fourier-truncated, GP equation, even though its analogue has been investigated in the 3D case (see references in [1]). Illustrative plots of the compressible kinetic-energy spectra are shown in Fig. 1.

References

[1] V. Shukla, M. Brachet, and R. Pandit. Turbulence in the two-dimensional fourier-truncated gross-pitaevskii equation. *arXiv*, **arXiv:1301.3383**, 2013.