# MEAN FIELD MODEL FOR TURBULENCE TRANSITION IN PLANE POISEUILLE FLOW

Michael Rath[1] & Bruno Eckhardt [1]
[1]*Fachbereich Physik, Philipps-Universität Marburg, Renthof 6, D-35032 Marburg, Germany*

*Abstract* In the pipe flow model of Dwight Barkley [1, 2], the main idea is to model pipe flow as an excitable, bistable medium. Using a one-dimensional FitzHugh-Nagumo-type reaction-advection-diffusion system with two variables the model captures qualitatively a surprising number of features of the turbulence transition in pipe flow. Motivated by this success, we here describe a derivation of a set of two 1+1-dimensional coupled differential equations for the closely related system of plane Poiseuille flow from the Navier-Stokes equation. The model contains terms for the production of turbulent kinetic energy, its transfer between the modes and its dissipation by viscous terms. The model shows a bifurcation to a non-trivial state and reflects some of the complex dynamics observed in direct numerical simulations.

## INTRODUCTION

The transition to turbulence in plane Couette flow, pipe flow and plane Poiseuille flow can be triggered by suitable perturbations in a range of Reynolds numbers where the laminar profile is still stable [6]. Considerable progress has been made through the identification of three-dimensional coherent structures [3, 5, 9, 11] that appear at flow rates below the ones where transition can be observed and which provide a scaffold of persistent structures around which the turbulence can form. The structures are remarkably consistent in appearance, are dominated by downstream vortices and can be rationalized within the self-sustaining cycle [7, 8]. They are exact solutions to the Navier-Stokes equation, and found numerically using suitably adapted and expanded Newton methods [10]. As an alternative to these exact methods we here discuss a more phenomenological approach that captures key elements of the structures without covering every technical detail. One of these key observations is that the turbulent flows have to be fully three-dimensional, with all velocity components active and variations in all three directions. If the flows are translationally invariant, then a straightforward analysis of the energy balance in the Navier-Stokes equation shows that there is no source of energy for the transverse velocity components so that they have to decay, and once they are gone also the modulation of the downstream velocity vanishes. Therefore, almost all studies of lifetimes and edge states use the energy content in the transverse velocity components as an indicator and not the total turbulent kinetic energy. Building on these observations, we here analyse the energy budget of the downstream and transverse velocity components, identify the transfer term, and provide a phenomenological model in terms of mean energy densities and transfer terms that gives the turbulence transition as a subcritical bifurcation.

## TOTAL ENERGY BALANCE

We start with the incompressible Navier-Stokes equation and decompose the velocity field into a base flow $\mathbf{u}_0$ and a perturbation $\mathbf{u}'$. For the two components of the model we take the downstream and transversal components, measured by their energy content. Therefore, we project the Navier Stokes equation for the perturbation onto the parallel $\mathbf{u}'_\parallel$ and perpendicular $\mathbf{u}'_\perp$ components, to arrive at a set of two coupled equations:

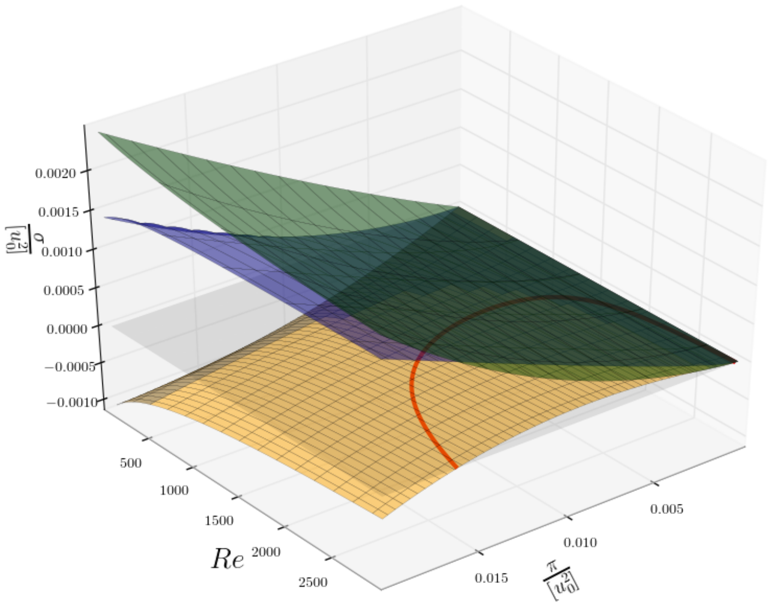
$$\partial_t \frac{\mathbf{u}_\parallel^2}{2} + \underbrace{\nabla \cdot (\mathbf{u}' \frac{\mathbf{u}_\parallel^2}{2}) + \partial_\parallel u_0 \frac{\mathbf{u}_\parallel^2}{2}}_{advection} \underbrace{+\mathbf{u}_\parallel (\mathbf{u}_\perp \cdot \partial_\perp) \mathbf{u}_0}_{production} = \underbrace{-\mathbf{u}_\parallel \cdot \nabla_\parallel p'}_{+transfer} + \underbrace{\frac{1}{Re} \mathbf{u}_\parallel \Delta \mathbf{u}_\parallel}_{dissipation_\parallel}$$

$$\partial_t \frac{\mathbf{u}_\perp^2}{2} + \underbrace{\nabla \cdot (\mathbf{u}' \frac{\mathbf{u}_\perp^2}{2}) + \partial_\parallel u_0 \frac{\mathbf{u}_\perp^2}{2}}_{advection} = -\nabla \cdot (\mathbf{u}' p') \underbrace{+\mathbf{u}_\parallel \cdot \nabla_\parallel p'}_{-transfer} + \underbrace{\frac{1}{Re} \mathbf{u}_\perp \Delta \mathbf{u}_\perp}_{dissipation_\perp}$$

Assuming a steady state and averaging over the entire volume gives a simple balance between energy production $P$, the transfer $T$ between the modes and the dissipations in the parallel and perpendicular components,

$$\begin{aligned} P &= T + \epsilon_\parallel \\ T &= \epsilon_\perp \end{aligned}$$

We can then express these terms as functions of the two variables $\pi := \langle \frac{\mathbf{u}_\parallel^2}{2} \rangle$ and $\sigma := \langle \frac{\mathbf{u}_\perp^2}{2} \rangle$ by parameter estimation from DNS data on fully turbulent flows, obtained using channelflow [4]. This results in a set of equations with two unknowns describing the nullclines $\dot{\pi} = 0$ (eq.(1)) and $\dot{\sigma} = 0$ (eq.(2)) of the system in a homogenous, steady state:



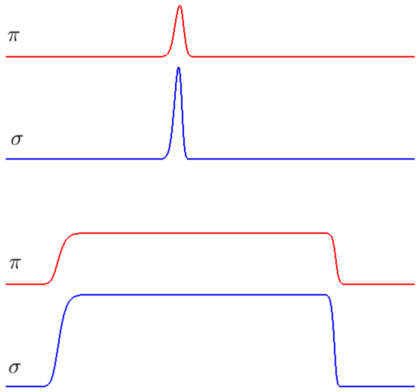




**Figure 2.** Localized puff solution for $Re = 1425.36$ (top pair) and growing slug solution for $Re = 1600$. (bottom pair)

**Figure 1.** Nullclines $\dot{\pi} = 0$ (green), $\dot{\sigma} = 0$ (blue) and their difference (beige). The saddle node states are shown as a projection onto the $\pi Re$ plane (red line)

$$(a_P + \frac{b_P}{Re})\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}} = (a_T + \frac{b_T}{Re})\pi\sigma^{\frac{1}{2}} + \frac{a_\epsilon^{\parallel}}{Re}\pi + b_\epsilon^{\parallel}\pi^{\frac{3}{2}} \tag{1}$$

$$(a_T + \frac{b_T}{Re})\pi\sigma^{\frac{1}{2}} = \frac{a_\epsilon^{\perp}}{Re}\sigma + b_\epsilon^{\perp}\sigma^{\frac{3}{2}}, \tag{2}$$

This set of equations has as fixed points the laminar state, $\pi = \sigma = 0$, and, as the Reynolds number increases, two new states created in a saddle node bifurcation at $Re \approx 1381$ (see the null clines in Figure 1).
When averaged over planes perpendicular to the flow direction, the downstream direction remains as a variable, and the advection and diffusion terms contribute partial derivatives. This then gives the final advection-diffusion equation

$$\partial_t\pi + a_{adv}^{\parallel}\partial_x\pi + \tilde{P}(Re,\pi,\sigma) = T_\pi(Re,\pi,\sigma) + \tilde{\epsilon}^{\parallel}(Re,\pi) + \frac{1}{Re}\partial_{xx}\pi$$

$$\partial_t\sigma + a_{adv}^{\perp}\partial_x\sigma + T_\sigma(Re,\pi,\sigma) = \tilde{\epsilon}^{\perp}(Re,\sigma) + \frac{1}{Re}\partial_{xx}\sigma$$

Examples of its solutions are shown in Figure 2.

## References


[1] D. Barkley. Modeling the transition to turbulence in shear flows. *Journal of Physics: Conference Series*, **318**:032001, 2011.
[2] D. Barkley. Simplifying the complexity of pipe flow. *Phys. Rev. E*, **84**:016309, Jul 2011.
[3] Holger Faisst and Bruno Eckhardt. Traveling waves in pipe flow. *Phys. Rev. Lett.*, **91**:224502, Nov 2003.
[4] J. F. Gibson. Channelflow: A spectral Navier-Stokes simulator in C++. Technical report, U. New Hampshire, 2012. `Channelflow.org`.
[5] J. F. Gibson, J. Halcrow, and P. Cvitanović. Visualizing the geometry of state space in plane Couette flow. *J. Fluid Mech.*, **611**:107–130, 2008.
[6] Siegfried Grossmann. The onset of shear flow turbulence. *Rev. Mod. Phys.*, **72**:603–618, Apr 2000.
[7] J. M. Hamilton, J. Kim, and F. Waleffe. Regeneration mechanisms of near-wall turbulence structures. *Journal of Fluid Mechanics*, **287**:317–348.
[8] G. Kawahara and S. Kida. Periodic motion embedded in plane couette turbulence. *J. of Fluid Mech.*, **449**:291–300, 2001.
[9] G. Kawahara, M. Uhlmann, and L. van Veen. *Ann. Rev. of Fluid Mech.*, **44**(1):203–225, 2012.
[10] Y. Duguet C. C. T. Pringle R. R. Kerswell. Relative periodic orbits in transitional pipe flow. *Physics of Fluids*, **20**(11):114102, 2008.
[11] B. Eckhardt T. M. Schneider B. Hof J. Westerweel. Turbulence transition in pipe flow. *Ann. Re. of Fluid Mechanics*, **39**(1):447–468, 2007.