DESCRIPTION OF TURBULENT RAYLEIGH–BÉNARD CONVECTION BY PDF METHODS EXHIBITS LIMIT CYCLE BEHAVIOR

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Abstract Rayleigh–Bénard convection, i.e. the convection of a fluid enclosed between two plates that is driven by a temperature gradient, is the idealized setup of a phenomenon ubiquitous in nature and technical applications. Of special interest for this system are the statistics of turbulent temperature fluctuations, which we are investigating for a fluid enclosed in a cylindrical vessel. To this end, we derive an exact evolution equation for the probability density function (PDF) of temperature from first principles. Appearing unclosed terms are expressed as conditional averages of velocities and heat diffusion, which are estimated from direct numerical simulations. Our theoretical framework allows to connect the statistical quantities to the dynamics of Rayleigh–Bénard convection, giving deeper insights into the temperature statistics and transport mechanisms in different regions of the fluid volume, i.e. in the boundary layers, the bulk and the sidewall regions.

STATISTICAL METHODS

Rayleigh–Bénard (RB) convection is the idealized fluid movement between two horizontal plates that is induced by a temperature gradient between the hot, lower and the cold, upper plate. This simple setup is the root of many convection phenomena appearing in natural and engineering systems, with examples including plate tectonics, atmospheric and oceanic flows, convection in the outer layers of stars, mixing of reagents in chemical reactors etc. All of the aforementioned examples exhibit turbulent fluid flows that are hard to describe, mainly due to the nonlinear, chaotic interactions; nevertheless, because of the broad applicability of the Rayleigh–Bénard system, a deeper understanding of the ongoing processes is desired even in the turbulent case. We refer to [1] for a recent review.

Because of the erratic and chaotic fluid movement (cf. the tangled structure of the temperature field in figure 1), a pointwise description of the system is not achievable, so we want to achive a statistical description of turbulent Rayleigh–Bénard convection by examining and describing the statistics of temperature fluctuations. Our ansatz starts from first principles and connects the statistics (in form of the probability density function (PDF) of temperature) to the dynamics of the system.

The starting point of our ansatz are the basic equations of the system, the Oberbeck-Boussinesq equations for velocity field $\boldsymbol{u}$ and temperature field T:

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \mathrm{Pr}\Delta\boldsymbol{u} + \mathrm{Pr}\,\mathrm{Ra}\,T\boldsymbol{e}_z\ ,\ \nabla\cdot\boldsymbol{u} = 0 \quad (1)$$

$$\frac{\partial}{\partial t}T + \boldsymbol{u}\cdot\nabla T = \Delta T \quad (2)$$

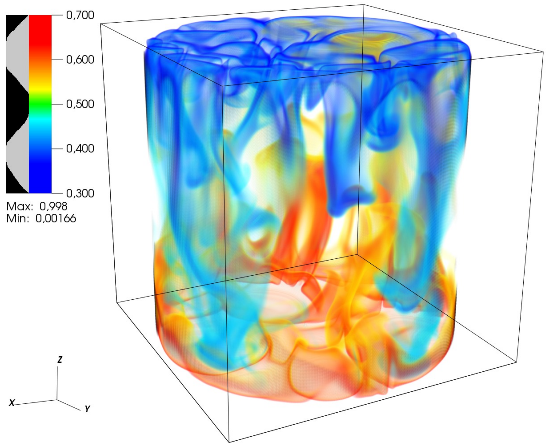


Figure 1: Volume rendering of temperature field at parameters $\mathrm{Ra} = 2\times10^8$, $\mathrm{Pr} = 1$.

To link the basic equations to a statistical description, we define the temperature PDF as $f(T,\boldsymbol{x},t) = \langle\delta\big(T - T(\boldsymbol{x},t)\big)\rangle$ and apply what is often referred to as *PDF methods* [4]. We follow the basics steps outlined in [3] and calculate temporal and spatial derivatives of $f(T,\boldsymbol{x},t)$, put them together, introduce conditional averages and plug in the Oberbeck-Boussinesq equations to obtain an evolution equation that describes the shape and deformation of the temperature PDF:

$$\frac{\partial}{\partial t}f + \nabla\cdot\big(\langle\boldsymbol{u}|T,\boldsymbol{x},t\rangle f\big) = -\frac{\partial}{\partial T}\big(\langle\Delta T|T,\boldsymbol{x},t\rangle f\big) \quad (3)$$

Here, unclosed terms in the form of conditional averages $\langle\cdot|T,\boldsymbol{x},t\rangle$ show up that are later estimated from direct numerical simulations (DNS) of RB convection in a cylinder. Due to the azimuthal symmetry of convection in a cylinder and due to stationarity in time, the dependence of the PDF becomes $f(T,\boldsymbol{x},t) = f(T,r,z)$ (with r and z being the radial and axial cylindrical coordinates, resp.), and also the spatial derivatives in (3) simplify and the time-derivative vanishes, which leads to the PDF-defining equation

$$\frac{1}{r}\frac{\partial}{\partial r}\big(r\langle u_r|T,r,z\rangle f\big) + \frac{\partial}{\partial z}\big(\langle u_z|T,r,z\rangle f\big) = -\frac{\partial}{\partial T}\big(\langle\Delta T|T,r,z\rangle f\big). \quad (4)$$

This is a first-order partial differential equation that can be analyzed with the Method of Characteristics [2]. Doing so, one obtains trajectories (the so-called *characteristics*) through the phase space spanned by the remaining three variables (T, r and z) by integrating the conditional averages:

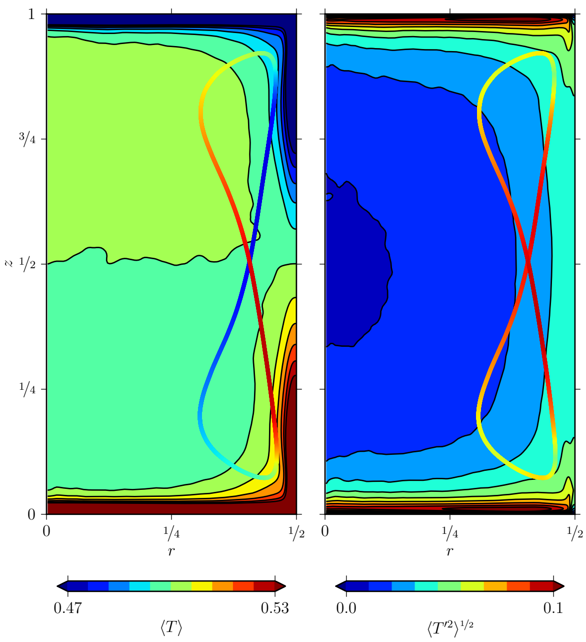
$$\dot{T} = \langle \Delta T | T, r, z \rangle \ , \ \dot{r} = \langle u_r | T, r, z \rangle \ , \ \dot{z} = \langle u_z | T, r, z \rangle \tag{5}$$

This means that the characteristics follow the conditionally averaged vector field through phase space and show the average evolution of a family of fluid parcels defined by their initial conditions; therefore, the characteristics show the *average fluid transport in phase space*.

RESULTS FROM NUMERICS

The three conditional averages $\langle \cdot | T, r, z \rangle$ that define the characteristics (5) are unclosed terms, which means they have to be specified externally. Our ansatz is to estimate them from Eulerian snapshots of DNS simulations; we simulated the basic equations for $\mathrm{Ra} = 2 \times 10^8$, $\mathrm{Pr} = 1$ in a cylindrical geometry by discretizing them on a staggered grid [5].

When integrating the vector field of the characteristics (5) for arbitrary starting points (T_0, r_0, z_0), one observes a convergence of all characteristics to a unique *limit cycle*. This limit cycle is depicted as the slender figure-of-eight form in figure 2(a) (or, to be precise, a projection into the r-z plane with the T-dimension color-coded). Tracing this limit cycle, one can reconstruct the typical *Rayleigh–Bénard cycle*: Starting at the lowest point of the figure-of-eight, one can see that fluid heats up fast near the bottom plate, starts to rise up and move inwards, move outwards and cool down at the top plate and fall down while moving inwards again. Additionally, the mean temperature field is shown in the background of figure 2(a), so one can compare the temperature deviations between fluid parcels traveling along the limit cycle and the mean temperature profile, thus indicating the regions of buoyancy. The regions of strongest buoyancy can also be seen in figure 2(b), where the deviation of the limit cycle temperature from the mean temperature is compared to the standard deviation of the temperature field in the respective regions; one can see that the standard deviation of the temperature field (shown in the background of figure 2(b)) is much lower than the deviation of the limit cycle temperature from the mean temperature. Thus, one can identify the bulk regions as the regions of strongest vertical movement due to buoyancy, while the main movement in temperature direction (i.e., cooling and heating) and in radial direction takes place in the boundary layers close to the top and bottom plates. This shows how our ansatz is able to connect the statistics to the dynamics of the RB system.



(a) Temperature (b) Temperature deviation

Figure 2: Limit cycle that shows up in the characteristics. (a) Mean temperature (background) and temperature of the limit cycle (figure-of-eight). (b) Standard deviation of temperature (background) and absolute deviation of the temperature of the limit cycle from the background temperature (figure-of-eight).

The reasons for and the impact of the existence of this limit cycle are currently being examined, e.g.: What are the consequences for the temperature PDF and its moments? What is the deeper meaning of the appearance of a limit cycle? These and more questions will be worked out in the near future.

SCOPE OF THE TALK

In the talk, I will briefly introduce into the topic of RB convection and PDF methods. Then I will show the numerical results and discuss the aforementioned limit cycle and its properties, with a focus on a connection to the dynamics of the system (i.e., discuss additional conditionally averaged flow fields in the form of movies, show flow patterns in different regions of the phase space etc.). In the end, I will link the dynamics of the system back to its statistics and discuss implications of the finding of a limit cycle for the statistical quantities, and give a short overview of the next steps to undertake.

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