COMPLETE CLASSIFICATION OF DISCRETE RESONANT ROSSBY/DRIFT WAVE TRIADS ON PERIODIC DOMAINS

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Abstract We consider the set of Diophantine equations that arise in the context of the partial differential equation called "barotropic vorticity equation" on periodic domains, when nonlinear wave interactions are studied to leading order in the amplitudes. The solutions to this set of Diophantine equations are of interest in atmosphere (Rossby waves) and Tokamak plasmas (drift waves), because they provide the values of the spectral wavevectors that interact resonantly via three-wave interactions. These wavenumbers come in "triads". We provide the full solution to the Diophantine equations in the physically sensible limit when the Rossby deformation radius is infinite. The method is completely new, and relies on mapping the unknown variables via rational transformations, first to rational points on elliptic curves and surfaces, and from there to rational points on quadratic forms of "Minkowski" type (such as the familiar *space-time* in special relativity). Classical methods invented centuries ago by Fermat, Euler, Lagrange, Minkowski, are used to classify all solutions to our original Diophantine equations, thus providing a computational method to generate numerically all the resonant triads in the system. Computationally speaking, our method has a clear advantage over brute-force numerical search: on a 10000^2 grid, the brute-force search would take 15 years using optimised C^{++} codes on a cluster, whereas our method takes about 40 min using a laptop. The method is extended to generate so-called quasi-resonant triads, which are defined by relaxing the resonant condition on the frequencies, allowing for a small mismatch. Quasi-resonant triads' distribution in wavevector space is robust with respect to physical perturbations, unlike resonant triads' distribution. Therefore, the extended method is really valuable in practical terms. We show that the set of quasi-resonant triads form an intricate network of connected triads, forming clusters whose structure depends on the value of the allowed mismatch. It is believed that understanding this network is absolutely relevant to understanding turbulence. We provide some quantitative comparison between the clusters' structure and the onset of fully nonlinear turbulent regime in the barotropic vorticity equation, and we provide perspectives for new research.

Introduction. Understanding the behaviour of nonlinear wave resonances is of importance in numerical weather prediction, fusion reactors and in general in experiments in fibre optics and water waves. Modulational instability is the first example of relevance of resonant and quasi-resonant triads and quartets [1]. Effects such as wave turbulence with its cascades of energy and enstrophy rely on the interconnected web of resonances. However a thorough understanding of the full-scale mechanism of energy transfers is lacking at the moment and one of the reasons for this is the poor understanding of the web of resonant and quasi-resonant triads. We will concentrate on the kinematical aspect (i.e., regarding the actual position in wavenumber space of the modes that interact in resonant and quasi-resonant triads), in the context of the barotropic vorticity equation on periodic domains.

Rossby Triads. We consider the dynamics of a rotating shallow layer of incompressible fluid, in the so-called β-plane approximation. The governing equation reduces to the following partial differential equation, known as the barotropic vorticity equation:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + \left(\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right) + \beta\frac{\partial\psi}{\partial x} = 0, \tag{1}$$

where $\psi = \psi(x, y, t) \in \mathbb{R}$ is the streamfunction, β is a constant that determines the speed of rotation of the system, and $F = 1/R^2$ where R is the Rossby deformation radius. For technical reasons we consider from here on the physically sensible limit of infinite Rossby deformation radius, $F = 0$. This means that we are looking at small-scale structures. We remark that the triads we will find still have a range of applicability into the more realistic case $F > 0$.

Equation (1) is also known in the literature as the Charney-Hasegawa-Mima equation, or CHM. The "C" comes from the atmospherical context just described. The "HM" comes from an independent derivation of the same mathematical equation in the context of plasma physics [7]. Due to this multidisciplinary aspect, the CHM equation occupies a special place in physics and mathematics.

The nonlinear term in Eq.(1) is responsible for the mixing of modes and is the starting point in the construction of so-called resonant triad solutions, which are approximate solutions, valid in the asymptotic limit when the oscillation amplitudes are small. Without going into the details of the multiple-scales method [9], these solutions are linear combinations of three travelling waves: $\Phi(x, y, t) = \Re\left(A_1(t)\Psi_{(k_1,l_1)}(x, y, t) + A_2(t)\Psi_{(k_2,l_2)}(x, y, t) + A_3(t)\Psi_{(k_3,l_3)}(x, y, t)\right)$, where $\Re$ denotes real part, $\Psi_{(k,l)}(x, y, t) \equiv e^{i(k\,x+l\,y-\omega(k,l)\,t)}$, $\omega(k, l) \equiv -\frac{\beta\,k}{k^2+l^2}$, A_j are some complex functions of time that are "slow" compared to the waves, and the set of wave vectors satisfy the following system of equations:

$$k_1 + k_2 = k_3\,, \qquad l_1 + l_2 = l_3\,, \qquad \omega_1 + \omega_2 = \omega_3\,, \tag{2}$$

where $\omega_j \equiv \omega(k_j, l_j)\,, \quad j = 1, 2, 3$. Any set of three wavevectors satisfying equations (2) is called a resonant triad. A consistency condition leads to a nonlinear system of evolution equations for the three slow functions A_j. We refer the reader to the vast literature on the subject, such as [9, 5] and references therein. In a more general setting, these slow functions also depend on the space coordinates [10], again 'more slowly' than the travelling waves.

It is possible to extend the resonant triad solutions to a more general approximate solution, consisting of a combination of groups of resonant triads: $\Phi(x,y,t) = \Re \sum_{I=1}^{N} A_I(t) \Psi_{(k_I,l_I)}(x,y,t)$, in such a way that any wavevector (k_I, l_I) appearing in the combination above, belongs to at least one resonant triad. In this setting, several resonant triads may be connected, forming so-called clusters. The main question of this paper is the classification of all possible values of wavevectors (k_I, l_I) that belong to resonant and near-resonant triads when the spatial domain is periodic. However, it is worth mentioning that the N functions $A_I(t)$ satisfy a coupled nonlinear system of evolution equations, that is derived using a set of consistency conditions, in a straightforward manner [8, 2].

Periodic Domains, Numerical Simulations and Discrete Resonant Triads. Periodic spatial domains are the common place for numerical simulations. One of the most reliable existing methods for periodic domains is the pseudo-spectral method, that takes advantage of the fast Fourier transform to compute partial derivatives and nonlinear terms. The pseudo-spectral method is widely used in numerical experiments due to its rapid convergence in terms of accuracy. In this method, solutions to equation (1) are sought on the periodic domain $(x,y) \in [0,2\pi) \times [0,2\pi)$, so solutions $\psi(x,y,t)$ satisfy $\psi(x,y,t) = \psi(x+2\pi,y,t) = \psi(x,y+2\pi,t)$. The numerical method begins by approximating the solution $\psi(x,y,t)$ as a finite sum of Fourier modes: $\psi(x,y,t) = \Re \sum_{k=0}^{N_x} \sum_{l=-N_y}^{N_y} A_{(k,l)}(t) \Psi_{(k,l)}(x,y,t)$. The unknown coefficients $A_{(k,l)}(t)$ need to be advanced in time numerically, using high-order techniques such as 4^{th}-order Runge-Kutta. The natural numbers N_x, N_y denote the spatial resolution of the numerical approximation, so that oscillations of wavenumber greater than these numbers cannot be resolved by the numerical scheme.

In the periodic setting, the relevant wavevectors (k,l) belong to a discrete lattice of integer numbers, so the resonant conditions (2) are now a system of Diophantine equations. For Rossby waves, the dispersion relation $\omega(k,l) \equiv -\frac{\beta k}{k^2+l^2}$ applies, and after some algebra the resonant conditions can be reduced to:

$$k_3(k_1^2+l_1^2)^2 - 2k_3(k_1^2+l_1^2)(k_3\,k_1+l_3\,l_1) + 2k_1(k_3\,k_1+l_3\,l_1)(k_3^2+l_3^2) - k_1(k_3^2+l_3^2)^2 = 0\,, \tag{3}$$

which can be interpreted as an equation for the pair (k_1, l_1), if the pair (k_3, l_3) is given. Here, the pair (k_2, l_2) is obtained a posteriori by writing $(k_2, l_2) = (k_3, l_3) - (k_1, l_1)$.

Classification of solutions of triad equations in terms of Fermat's theorem of sums of squares. After a number of rational transformations the resonant condition (3) can be mapped to rational points in elliptic curves:

$$X^3 - 2\,D\,X^2 + 2\,D\,X - D^2 = Y^2\,. \tag{4}$$

See full details in [3]. The mapping is bijective up to overall re-scaling of the triad wavenumbers, and given explicitly by $X = \frac{k_3}{k_1} \times \frac{k_1^2+l_1^2}{k_3^2+l_3^2}$, $\quad Y = \frac{k_3}{k_1} \times \frac{k_3 l_1 - k_1 l_3}{k_3^2+l_3^2}$, $\quad D = \frac{k_3}{k_1} \times \frac{k_3 k_1 + l_3 l_1}{k_3^2+l_3^2}$, and with inverse $\frac{k_1}{k_3} = \frac{X}{D^2+Y^2}$, $\quad \frac{l_1}{k_3} = \frac{X}{Y}\left(1 - \frac{D}{D^2+Y^2}\right)$, $\quad \frac{l_3}{k_3} = \frac{D-1}{Y}$, and $(k_2, l_2) = (k_3 - k_1, l_3 - l_1)$.

The elliptic curve (4) can be reduced to quadratic forms by defining $X \equiv -\frac{m+n}{m-n}, \quad m,n \in \mathbb{Z}$, so we get the equation: $\left(\frac{Y(m-n)^2}{m+n}\right)^2 + \left(\frac{D(m-n)^2}{m+n} + 2\,m\right)^2 = 3\,m^2 + n^2$. This equation is to be solved for $m, n, \in \mathbb{Z}$ and $Y, D \in \mathbb{Q}$. The cases $Y = 0$, $m = 0$ or $m = \pm n$ are excluded from the solutions: they give rise to zonal modes. Notice that since m, n are integers, the two members of this quadratic equation are equal to an integer. The problem of finding all possible integers that can be written as a sum of squares of two rationals is well known and dates back to Fermat [6, Chapter V] (it is called Fermat's Xmas theorem). Also, the problem of finding all possible integers in the form $3\,m^2 + n^2$ was considered by Fermat and solved by Lagrange and Euler, see [4, Chapter 1] for further details. The current equation is a combination of these two problems and can be dealt with in a straightforward manner. As a result, all possible representations for the numbers m, n and the numbers Y, D can be obtained explicitly. Consequently, all possible exact resonant triads can be obtained explicitly by using the inverse mapping from X, Y, D to the triad's wavenumbers.

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