REPRODUCTION OF 2D CHAOTIC ZONAL FLOW ON A ROTATING SPHERE

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Abstract We study the properties of chaotic solutions under steady zonal forcing at high Reynolds number using the steady solutions, which bifurcate from the basic zonal flow solution at low Reynolds number. We reproduce the zonal-mean zonal velocity of the chaotic solution from those of the unstable bifurcating solutions by making a linear mapping from the solution space to the zonal-mean zonal profiles. The reproduction of the zonal-mean profiles is satisfactory although the linear mapping assumes the linear inter- and extra-polation of the profile of the bifurcating solutions in the solution space.

INTRODUCTION

The two-dimensional incompressible Navier-Stokes flow on a rotating sphere is considered to be one of the simplest and most fundamental models of the atmospheric motions. Recently, Obuse et al. studied the forced turbulence on a rotating sphere, and they found that at the final stage of time integration, only two or three broad zonal jets are left in the flow field which are found to be quite *stable* to disturbance even in the ambient turbulent flows[1].
Sasaki et al. studied the linear stability of steady viscous zonal jet flows[2] . They found that, at the critical Reynolds number where the stability of zonal flow changes from stable to unstable with the Reynolds number increasing, the zonal flow becomes Hopf unstable. As the rotation rate increases, the critical Reynolds number increases rapidly and they found the discrepancy of the unstable region of the rotation rate between the stability of inviscid limit and the inviscid case[3]. This seeming contradiction is resolved by an observation that as the Reynolds number increases the growth rate of the unstable mode converges to zero in the regions where the viscous zonal flow is unstable while the inviscid zonal flow is stable. Here we study the bifurcation structure arising from the zonal jet flow and chaotic solutions at high Reynolds number.

GOVERNING EQUATION

We consider the two-dimensional incompressible viscous flow on a rotating sphere governed by the vorticity equation

$$\partial_t \Delta\psi + J(\psi, \Delta\psi) + 2\Omega\partial_\lambda \psi = \left\{ (\Delta + 2)\, \Delta\psi + (l(l+1) - 2)\, Y_l^0(\mu) \right\} / R,$$

where quantities are made non-dimensional and the radius of the sphere is unity. Here t is the time, λ and μ the longitude and the sine latitude $\mu = \sin\phi$ where ϕ is the latitude, ψ the streamfunction and $\Delta\psi$ the vorticity, where Δ is the horizontal Laplacian on the unit sphere. R and Ω are the Reynolds number and a non-dimensional rotation rate of the sphere, respectively. $J(A, B)$ is the Jacobian and $(l(l+1) - 2)Y_l^0(\mu)/R$ is the vorticity forcing where $Y_l^m(\lambda, \mu)$ is a 4π-normalized spherical harmonics with the total wavenumber l and the azimuthal wavenumber m. The vorticity equation has a steady l-jet zonal flow solution for any Reynolds number and any rotation rate, expressed by $\psi_0(\mu) = -\frac{1}{l(l+1)} Y_l^0(\mu)$. Here, the number of jets is defined as the number of extreme points of the longitudinal velocity, $u_\lambda(\lambda, \mu) = -\sqrt{1-\mu^2}\partial_\mu \psi(\lambda, \mu)$ We note that this problem setting is similar to the Kolmogorov problem on the point that the viscosity forcing is expressed by the eigenfunction of the Laplacian of each manifold, a sphere and a torus.

BIFURCATION OF STEADY SOLUTIONS AND CHAOTIC SOLUTION AT HIGH REYNOLDS NUMBER

The 1-jet zonal flow ($l = 1$) corresponds to the conservation of the total angular velocity and the 2-jet zonal flow ($l = 2$) is globally asymptotic stable[4]. Here we discuss the bifurcation structure arising from the 3-jet zonal flow ($l = 3$). Sasaki, Takehiro and Yamada studied the linear stability of the 3-jet zonal flow and they found that, in the non-rotating case as the Reynolds number increases the trivial solution becomes Hopf unstable at $R = 26.12$ with the critical longitudinal wavenumber $m_c = 2$[2]. As below we call the 3-jet zonal flow the trivial solution.
On the trivial solution branch we find two Hopf bifurcation points at $R = 26.12$ and $R = 62.51$. At each Reynolds number, a steady traveling wave solution bifurcates. Tracing each steady traveling solution branch, as the Reynolds number increases other steady traveling solutions bifurcate through the pitchfork bifurcation and these steady traveling solutions become Hopf unstable in the region of $249.4 \leq R \leq 10^4$. We find the eleven steady/steady traveling solutions at $R = 10^4$ including the trivial solution.
We carry out numerical time integrations at $R = 10^4$ and find that the energy of the unsteady solution undergoes intermittent bursts and its power spectrum is broad, that is, the unsteady solution is chaotic. The chaotic solution seems to wander around the unstable steady traveling solutions. To make clear the relationship between the chaotic solution and

the steady traveling solutions, we try to reproduce the zonal-mean zonal velocity of the chaotic solution by using those of the steady/steady traveling solutions. Here the zonal-mean zonal velocity is defined by $U(\mu) = \frac{1}{2\pi}\int_0^{2\pi} u_\lambda(\lambda,\mu)d\lambda$.
We consider a linear mapping defined as below; if the linear mapping operates each steady/steady traveling solution ψ_i the linear mapping gives its zonal-mean zonal velocity $U_i(\mu)$, while if the linear mapping operates the orthogonal compliment of the linear space which is extended by the steady/steady traveling solutions, the linear mapping gives zero. To obtain the function value of the linear mapping, consider a member of the linear space spanned by the steady/steady traveling solutions which expressed by a linear combination $\hat{U}(\mu,t) = \sum_i c_i(t)U_i(\mu)$. The coefficient $c_i(t)$ is obtained by the Hermite inner product $c_i(t) = \int \frac{d\mu}{2\pi} U_i^{\dagger *}(\mu)\hat{U}_i(\mu,t)$ where $U_i^\dagger(\mu)$ is the dual basis of $U_i(\mu)$ and * is the complex conjugate. When the zonal-mean zonal velocity of the chaotic solution for each instant of the time is substituted in $\hat{U}(\mu,t)$ of the Hermite inner product, the linear combination gives the function value of the linear mapping. Figure 1 shows the time-averaged reproduced zonal-mean zonal velocity by the linear mapping using the eleven steady/steady traveling solutions. We find that the linear mapping reproduce the zonal-mean zonal velocity of the chaotic solution for not only time-averaged but also each instant of the time. This result suggests that even at high Reynolds number, which is 40 times of the critical Reynolds number of the laminar flow, the chaotic solution lies mostly within a relatively low-dimensional space spanned by the steady/steady traveling solutions bifurcating from the trivial solution.

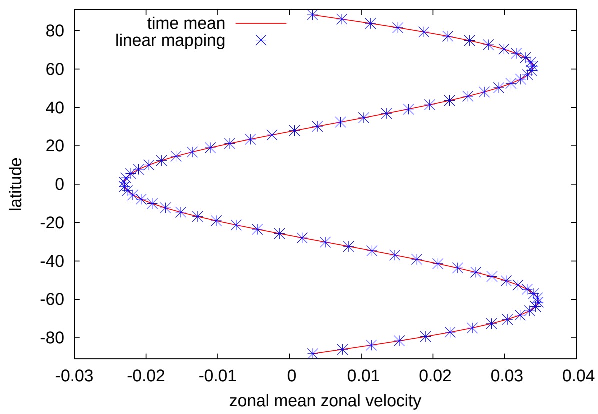

Figure 1. Reproduction of time-averaged zonal-mean zonal velocity at $R = 10^4$. The red line denotes time-averaged zonal-mean zonal velocity of the chaotic solution while the blue asterisks indicate the reproduced zonal-mean zonal velocity by the linear mapping.

CONCLUSION

In this paper we study the bifurcation structure arising from the 3-jet zonal flow and the properties of chaotic solutions at high Reynolds number.
As Reynolds number increases, the steady traveling solutions bifurcate from the 3-jet zonal flow through the Hopf bifurcation. As the Reynolds number further increases, the several traveling solutions arise only through the pitchfork bifurcation from the traveling solutions and finally the steady traveling solutions become Hopf unstable.
We carry out time integration at high Reynolds number. The unsteady solution is chaotic and seems to wander around the steady/steady traveling solutions. In order to find out the relationships between the chaotic solution and the steady/steady traveling solutions, we try to reproduce the zonal-mean zonal velocity of the chaotic solution using the unstable steady/steady traveling solutions bifurcating from the 3-jet zonal flow. Considering the linear mapping from solution space to the zonal-mean zonal profiles spanned by the steady/steady traveling solutions, we find that the linear mapping reproduces sufficiently well the zonal-mean zonal velocity of the chaotic solution for each instant of the time. This result suggests that even at high Reynolds number the chaotic solutions exist on the linear space spanned by the unstable solutions bifurcating from 3-jet zonal flow at low Reynolds number.

References

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