SCALING LAWS FOR CONVECTIVE DYNAMOS

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Abstract We study magnetic dynamos in developed (turbulent) convection. The scaling laws for the strength of the induced magnetic field measured by the Hartmann and the Nusselt number (measuring the superadiabatic convective heat flux) number with the Rayleigh number (measuring the strength of the driving force) are found. Three cases are studied - two of them covering the range of parameters achievable in numerical and possible future laboratory experiments and the third corresponding to the parameter regime for the Earth's liquid outer core. We find, that at least in the case of Boussinesq dynamos the presence of induced magnetic field tends to decrease (or leave unchanged) the order of magnitude of the convective heat flux in the system. The effect of compressibility is also studied; it is found to be more complex and it typically decreases the convective heat flux.

INTRODUCTION

The aim of this paper is to combine current ideas on the saturation of turbulent dynamos summarised in [1] with those concerning the developed turbulent convection explained in the comprehensive study of Grossman and Lohse [2]. The scaling laws for the convective heat flux (measured by the Nusselt number) and the magnitude of the convective velocities (measured by the Reynolds number) with the Rayleigh number obtained by [2] were generalised to the case of developed magnetoconvection, i.e. convection with an imposed vertical magnetic field, by [3] and [4]. When the external magnetic field is given beforehand, the Hartmann number M - a dimensionless measure of the strength of the magnetic field, is an input parameter and the theory typically predicts scaling laws of the form $(Nu, Re) \sim Ra^{\beta_1}/M^{\beta_2}$.

In magnetic dynamos the magnetic field is solely due to the vigorous convective motions and the Hartmann number is a part of the solution, hence is unknown. It is of interest from the point of astrophysical and geophysical studies, to provide the scaling laws for the magnitude of the induced magnetic field and the Nusselt number as a function of Rayleigh number in convecting systems. To obtain such scaling laws, we need the relation between the kinetic and magnetic energies for saturated dynamos. There are two clear cases, identified by [5] and [6] which are relevant to geophysics and astrophysics, for which the process of dynamo saturation has been discussed. The first regime, applied to liquid metals, stellar interiors or accretions discs is obtained for small and moderate magnetic Prandtl numbers, $Pm < 1$, and corresponds to equipartition between the kinetic and magnetic energies,

$$\frac{B^2}{\mu_0} \sim \rho u^2, \qquad (Pm < 1). \tag{1}$$

The second, corresponding to galaxies and galaxy clusters is defined by $1 \ll Pm \leq Re^{1/2}$, when at the saturated level the kinetic and magnetic energies are not equally partitioned and (cf. [5])

$$\frac{B^2}{\mu_0} \sim \rho u^2 Re^{-1/2} Pm, \qquad \left(1 \ll Pm \leq Re^{1/2}\right). \tag{2}$$

In the above ρ is the density of the fluid, μ_0 is the magnetic permeability of vacuum, u and B are the magnitudes of the convective velocity and induced magnetic field respectively, $Pm = \nu/\eta$, $Re = ud/\nu$ where ν and η are the kinematic viscosity and the magnetic diffusivity respectively.

The system we consider is a layer of fluid confined between two horizontal plates at distance d, periodic in horizontal directions with gravity $\mathbf{g}$ pointing downwards along the z axis. This is the configuration often used in numerical experiments. No laboratory convection driven dynamos exist, but convection with an imposed field has been studied and laboratory dynamos forced by impellers are known. Standard no-slip conditions are imposed at the boundaries, which are impermeable and either conducting or insulating. The fluid is heated from below and the boundaries are held at constant temperatures.

The main objective is the derivation of the scaling laws for the Hartmann and Nusselt numbers with Ra in developed convection. The form of the scaling laws crucially depends on whether the thermal, viscous and magnetic dissipation take place in the boundary layers or in the bulk of the convection. According to the diagram given in [2] (their figure 2) for non-magnetic convection there are two regimes corresponding to currently achievable Rayleigh numbers ($Ra \lesssim 10^{13}$) and typically used Prandtl numbers ($10^{-6} \lesssim Pr = \nu/\kappa \lesssim 10$) in experiments, i.e. first when the viscous and thermal dissipation predominantly takes place in the boundary layers, and second when the viscous dissipation is dominated by its bulk contribution and the thermal by its boundary layer contribution. At Rayleigh numbers larger than can be achieved in laboratory convection, which may nevertheless be appropriate for astophysical and geophysical convection, the boundary layers become turbulent, and both viscous and thermal dissipation are dominated by their bulk contributions.

RESULTS

For each of the two equipartition regimes $Pm < 1$ and $Pm \gg 1$ there are four possible combinations depending on whether the viscous and magnetic dissipation occurs primarily in the bulk or the boundary layers. We denote the case where both viscous and magnetic dissipations are dominated by their boundary layer contributions by V_{BL} M_{BL}, and when both are dominated by their bulk contribution by V_{BU} M_{BU}, and similarly for the mixed cases.
The scaling laws for Boussinesq dynamos in the experimental regimes are summarised in the Table 1.

	$Pm < 1$		$1 \ll Pm \le Re^{1/2}$	
	$\frac{\mu_0 \eta}{\rho} \langle \mathbf{j^2} \rangle / \nu \langle \nabla\mathbf{u} : \nabla\mathbf{u} \rangle$	Convective dynamo scaling laws	$\frac{\mu_0 \eta}{\rho} \langle \mathbf{j^2} \rangle / \nu \langle \nabla\mathbf{u} : \nabla\mathbf{u} \rangle$	Convective dynamo scaling laws
$\mathrm{V}_{BU}\mathrm{M}_{BU}$	1	$M \sim Ra^{2/5}Pr^{-3/5}Pm^{1/2}$ $Nu \sim Ra^{1/5}Pr^{1/5}$ $Re \sim Ra^{2/5}Pr^{-3/5}$	$Re^{-1/2}Pm$	$M \sim Ra^{3/10}Pr^{-9/20}Pm$ $Nu \sim Ra^{1/5}Pr^{1/5}$ $Re \sim Ra^{2/5}Pr^{-3/5}$
$\mathrm{V}_{BL}\mathrm{M}_{BU}$	$Pm^{-1/2}$	$M \sim Ra^{2/5}Pr^{-3/5}Pm^{1/2}$ $Nu \sim Ra^{1/5}Pr^{1/5}$ $Re \sim Ra^{2/5}Pr^{-3/5}$	$Re^{-1/4}$	$M \sim Ra^{1/3}Pr^{-1/2}Pm^{2/3}$ $Nu \sim Ra^{2/9}Pr^{1/6}Pm^{-2/9}$ $Re \sim Ra^{4/9}Pr^{-2/3}Pm^{-4/9}$
$\mathrm{V}_{BU}\mathrm{M}_{BL}$	$Pm^{-1/2}$	$M \sim Ra^{2/5}Pr^{-3/5}Pm^{7/10}$ $Nu \sim Ra^{1/5}Pr^{1/5}Pm^{1/10}$ $Re \sim Ra^{2/5}Pr^{-3/5}Pm^{1/5}$	$Re^{-3/4}Pm$	$M \sim Ra^{3/10}Pr^{-9/20}Pm$ $Nu \sim Ra^{1/5}Pr^{1/5}$ $Re \sim Ra^{2/5}Pr^{-3/5}$
$\mathrm{V}_{BL}\mathrm{M}_{BL}$	Pm^{-1}	$M \sim Ra^{2/5}Pr^{-3/5}Pm^{7/10}$ $Nu \sim Ra^{1/5}Pr^{1/5}Pm^{1/10}$ $Re \sim Ra^{2/5}Pr^{-3/5}Pm^{1/5}$	$Re^{-1/2}$	$M \sim Ra^{1/3}Pr^{-1/2}Pm^{2/3}$ $Nu \sim Ra^{2/9}Pr^{1/6}Pm^{-2/9}$ $Re \sim Ra^{4/9}Pr^{-2/3}Pm^{-4/9}$

Table 1. The scaling laws in the experimental regimes. *Re* is the Reynolds number.

The following scaling laws have been obtained for the case of geophysical regime

$$M \sim Ra^{1/2}Pr^{-3/4}Pm^{3/4}u_*^{-1/2}2^{-1/4}, \tag{3a}$$

$$Nu \sim Ra^{1/2}Pr^{-1/4}Pm^{3/4}u_*^{-3/2}2^{-3/4}, \tag{3b}$$

$$Re \sim Ra^{1/2}Pr^{-3/4}Pm^{1/4}u_*^{-1/2}2^{-1/4}. \tag{3c}$$

where u_* is the dimensionless friction velocity (cf. [7]) appearing in the standard theory of turbulent boundary layer.
Moreover, the effect of compressibility under the anelastic approximation has been studied. It found that the total heat entering the system at the bottom (or leaving at the top) typically decreases with the compressibility, i.e. the smaller the density scale height the smaller the Nusselt number.

References

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