A NEW SPECTRAL METHOD FOR DIRECT NUMERICAL SIMULATIONS OF MAGNETOHYDRODYNAMIC CHANNEL FLOWS

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Abstract Simulations of liquid metal flows in channel and duct configurations under a strong magnetic field pose a difficult problem for existing numerical methods. The main obstacle is the linear increase in number of modes required to resolve thin Hartmann boundary layers with the intensity of the magnetic field B. To overcome this problem we developed a new approach to the numerical calculations describing these flows. The solution of the flow is expressed in a base of eigenfunctions of the linear part of the governing equations and its adjoint, We show that in this approach the computational cost does not depend on the thickness of boundary layer and therefore it allows for performing calculations for high magnetic fields.

EQUATIONS

We consider a flow of liquid metal in channel configuration with the homogeneous magnetic field and impermeable, electrically insulating walls located at $z = \pm 1$ in a low Rm regime. Under these assumptions the set of governing equations can be expressed in dimensionless form:

$$\frac{\partial \mathbf{u}}{\partial t} + P(\mathbf{u}\cdot\nabla)\mathbf{u} = \Delta\mathbf{u} - \frac{1}{Ha}\Delta^{-1}\partial_{zz}\mathbf{u} \tag{1}$$

where $Ha = LB\sqrt{\sigma/\rho\nu}$, called the Hartman number, represents the ratio of Lorentz to viscous forces and P denotes orthogonal projection onto the subspace of solenoidal fields.

NUMERICAL METHOD

We express the solution of eq. (1) in terms of eigenvectors of operator that represents linear part of eq. (1). In [1] a set of solenoidal solutions to eigenproblem of operator has been derived. Because these modes are obtained from an operator that reflects the physics of this class of flow, the set of modes built this way is made of elements that reflect structures of the actual flows. Therefore they are natural candidates to use as a base in a numerical spectral scheme. In particular, laminar and turbulent Hartmann boundary layers that develop along the channel walls appear as built-in features. Moreover it can be shown that to resolve the flow completely it is necessary to take into account all modes $\lambda < \lambda_{max}$ such as their number is equal to Re^2/Ha [1].

However in general the non linear terms cannot be expressed as an expansion in terms of the above modes, as they span only the divergence free subspace of all functions fulfilling the boundary conditions. Therefore our base has to be supplemented with additional elements spreading the irrotational subspace as well. We obtain them by solving the eigenproblem with condition that velocity field is solenoidal replaced with condition $\nabla\times\mathbf{u} = 0$.

Non linear terms

The main difficulty of solving equation (1) in spectral space lies in calculating non linear terms. We use a pseudospectral approach and calculate them in real space. Therefore we need a method to reconstruct a spectral coefficients g_n of physical vector fields known at the discrete set of points in space $\mathbf{x}_i$. This problem can be formulated as a set of linear equations for unknown spectral components:

$$g_n\mathbf{e}_n(\mathbf{x}_i) = \mathbf{G}(\mathbf{x}_i) \quad i = 1\ldots N \tag{2}$$

where $\mathbf{e}_n$ constitutes are base of eigenvectors, and $\mathbf{G}$ represents the decomposed vector field. As the coefficients in this set of equations are constant during a single numerical run, it is worth performing LU decomposition of the corresponding matrix at the beginning of calculations and later use it to efficiently find the spectral decompositions. Moreover it enables us to save even more CPU time by omitting calculation of coefficients which wouldn't be used in further calculations. For example we are interested only in the g_n coefficients corresponding to divergence free modes. Neglecting coefficients corresponding to irrotational modes is an equivalent of performing projection representing by operator P in eq. (1).

The technique described above has the advantage that the obtained spectral decomposition reproduces exactly the physical field on the given set of discretization points. Therefore momentum and energy are conserved by this procedure. However the obtained spectral coefficients g_n are different from the exact ones $\widetilde{g}_n$ that would be the result of decomposition of the same vector field in an infinite dimensional space spanned by all eigenvectors. The magnitude of the introduced error is shown in fig. (1). It shows the absolute value of difference between g_n and $\widetilde{g}_n$ for two values of n in function of number of modes used for decomposition N_z. Up to value $N_z = 300$ error is more or less constant as the used eigenmodes do

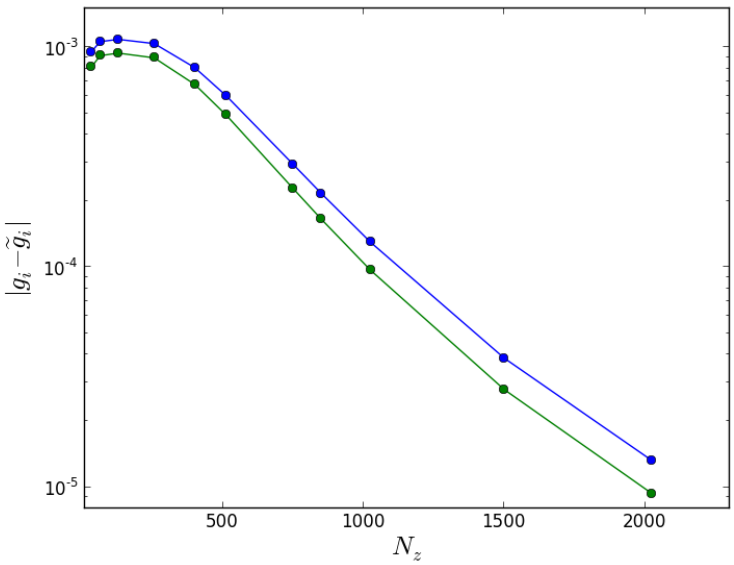

Figure 1. The absolute value of difference between g_n and $\widetilde{g}_n$ for two values of $n = 4, 12$ in function of grid size in z direction. The decomposed field is $(\mathbf{u} \cdot \nabla)\mathbf{u}$, where $\mathbf{u}$ is a single eigenmode for $Ha = 1000$ normalized in such a way the its amplitude is 1.

not resolve the Hartmann layer. Above this value of N_z error introduced by spectral decomposition starts to drop with exponential rate.

COMPUTATIONAL COST

The cost of pseudospectral code based on FFT technique is $N_x N_y N_z \log(N_x N_y N_z)$, where N_i denotes number of discretization points in direction i. In DNS calculations N_i should be of the order of Reynolds number $Re^{3/4}$ each. Moreover to obtain physically meaningful results it is necessary to resolve the Hartmann layer in z direction. Therefore N_z should be at least of the order of Ha. Therefore for large Harmann number calculations the computational cost of FFT based code scales as $\sim Re^{3/2} Ha \log(ReHa)$. For the problem considered in this contribution the dependence of the used eigenvectors on x and y is in the form of Fourier modes. Therefore the described in previous section spectral decomposition has to be used only for dependence on z coordinate, while spectral decomposition in $x - y$ can be performed with FFT. The computational cost in this case scales as $\sim N_x N_y N_z^2$. Using the estimation for the required number of modes from [2, 3]: $N_{x,y} \sim Re^{1/2}$, $N_z \sim Re/Ha$. However, based on fig. 1 we conclude that we still need to use sufficient number of modes ($\sim Ha$) in z directions to calculate accurate spectral coefficients. However we need to calculate only Re^2/Ha of spectral coefficients, as the rest of them will be dissipated on the very short timescales and are not physically meaningful. Therefore the total computational cost of the proposed scheme scales as $\sim Re^2$.

CONCLUSIONS

We argue that our new approach has a potential to significantly reduce the computational cost incurred by traditional methods for the calculation of thin dissipative layers as its cost does not depend on Ha. Therefore it should allow for performing calculations in physical regimes of very high magnetic field not currently accessible to numerical simulations.

References

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