ESTIMATION OF TURBULENCE-DEVELOPMENT BY A MULTIFRACTAL THEORY

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Abstract It is shown for the first time how the degree of development in turbulence, i.e., if it is fully developed or if not how much developed, can be estimated by the multifractal theory called *multifractal probability density function theory* (MPDFT) [1]. Precise and self-consistent analyses of 4096^3 DNS turbulence [2] by means of MPDFT reveal that the turbulence with $R_\lambda = 675$ is in the fully developed turbulent state, but that the one with $R_\lambda = 1132$ is *not* fully developed yet. The degree of turbulence-development of the latter is estimated.

BACKGROUND

It is known that, for small value ν, the Navier-Stokes (N-S) equation

$$\partial\vec{u}/\partial t + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}p + \nu\nabla^2\vec{u} \tag{1}$$

for an incompressible fluid ($\vec{\nabla}\cdot\vec{u} = 0$) is invariant under the scale transformation $\vec{x} \to \lambda\vec{x}$ with the rescaling of velocity field $\vec{u} \to \lambda^{\alpha/3}\vec{u}$, of time $t \to \lambda^{1-\alpha/3}t$ and of pressure $p \to \lambda^{2\alpha/3}p$ with an arbitrary real number α. Here, ν is the kinematic viscosity; $\vec{u}$ is the velocity field; p is the pressure of fluid per unit mass. In treating an actual turbulent system, the value of ν is fixed to a finite non-zero value unique to the fluid prepared for experiment. It is assumed that for high Reynolds number the singularities responsible for the turbulent motion of fluid distribute themselves in a multifractal way in real physical space [3]. This distribution of singularities may produce intermittent fluid motion of turbulence to which we refer as a *coherent* turbulent motion. The dissipation term $\nu\nabla^2\vec{u}$ in the N-S equation is interpreted as a term violating the invariance under the scale transformation. This violation may produce an *incoherent* fluctuating motion of fluid.

A&A model within MPDFT [1] assumes that the distribution of singularities is specified by Tsallis-type PDF [4], say $P^{(n)}(\alpha)$, for stochastic singularity exponents α, and that PDF $\Pi^{(n)}(\varepsilon_n)$ for the stochastic singular quantity such as the energy dissipation rates ε_n can be divided into two parts as

$$\Pi^{(n)}(\varepsilon_n) = \Pi^{(n)}_{\rm S}(\varepsilon_n) + \Delta\Pi^{(n)}(\varepsilon_n) \tag{2}$$

(see Fig. 1 (a) and (b)). The first term, describing the coherent turbulent motion, is specified by the Tsallis-type PDF through the relation [5]

$$\Pi^{(n)}_{\rm S}(\varepsilon_n)d\varepsilon_n = \bar{\Pi}^{(n)}_{\rm S} P^{(n)}(\alpha)d\alpha \tag{3}$$

(see [1] for the expression of $\bar{\Pi}^{(n)}_{\rm S}$). The second term represents the contribution from the incoherent fluctuating fluid motion. The stochastic energy dissipation rates

$$\varepsilon_n = \varepsilon_n(\alpha) = \epsilon_0(\ell_n/\ell_0)^{\alpha-1} \tag{4}$$

which reveal singular intermittent behavior in the limit $\ell_n/\ell_0 \to 0$ for $\alpha < 1$ are obtained by coarse-graining the microscopic dissipation rates inside the region of diameter ℓ_n. The normalizations of the PDFs are given by $\int_0^\infty d\alpha P^{(n)}(\alpha) = 1$ and $\int_0^\infty d\varepsilon_n \Pi^{(n)}(\varepsilon_n) = 1$.

Since turbulence is a typical system producing fat-tail PDFs for those physical quantities revealing singular behavior, we divide the fat-tail PDF $\Pi^{(n)}(\varepsilon_n)$ for energy dissipation rates into two parts, i.e., the *center* (cr) and *tail* (tl) parts,

$$\Pi^{(n)}(\varepsilon_n) = \Pi^{(n)}_{\rm cr}(\varepsilon_n) + \Pi^{(n)}_{\rm tl}(\varepsilon_n) \tag{5}$$

(see Fig. 1 (c)). The center part PDF $\Pi^{(n)}_{\rm cr}(\varepsilon_n)$ for $\varepsilon_n \le \varepsilon_n^*$ and the tail part PDF $\Pi^{(n)}_{\rm tl}(\varepsilon_n)$ for $\varepsilon_n \ge \varepsilon_n^*$ are connected at $\varepsilon_n = \varepsilon_n^*$. It is reasonable to assume that, for the tail part PDF $\Pi^{(n)}_{\rm tl}(\varepsilon_n)$, one can neglect in high precision the contribution from the second correction term in (2). Under this assumption, we put $\Delta\Pi^{(n)}(\varepsilon_n) = 0$ for $\varepsilon_n \ge \varepsilon_n^*$. To the center part PDF $\Pi^{(n)}_{\rm cr}(\varepsilon_n)$, both coherent and incoherent motions contribute (see [1] for details).

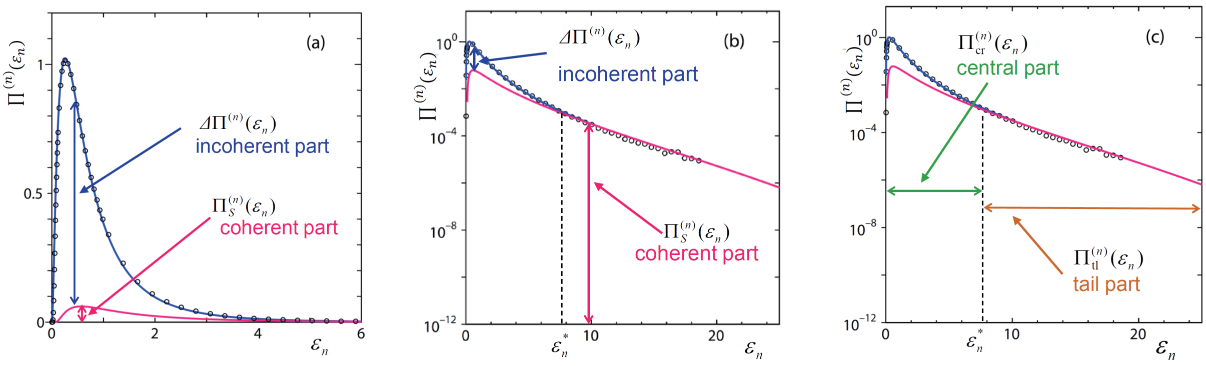

Figure 1. Two kinds of divisions of PDF $\Pi^{(n)}(\varepsilon_n)$. The division (2) is shown in (a) and (b), respectively, on linear and log scale in the vertical axes. Another division (5) is presented in (c) on log scale. ε_n^* is the connection point. The open circles represent an experimental PDF for energy dissipation rates. The contribution of $\varDelta\Pi^{(n)}(\varepsilon_n)$ to the tail part $\Pi_{\rm tl}^{(n)}(\varepsilon_n)$ is negligibly small.

ANALYSIS OF 4096^3 DNS

Since the stochastic quantities ε_n for energy dissipation rates are obtained by coarse graining the microscopic local dissipation rates in the region with diameter ℓ_n as mentioned before, the energy dissipation rate averaged with $\Pi^{(n)}(\varepsilon_n)$, i.e.,

$$\langle\!\langle \varepsilon_n \rangle\!\rangle = \int d\varepsilon_n\, \varepsilon_n \Pi^{(n)}(\varepsilon_n) \equiv \varepsilon, \tag{6}$$

is constant independent of ℓ_n. On the other hand, the mean value

$$\langle \varepsilon_n \rangle = \int d\alpha\, \varepsilon_n(\alpha) P^{(n)}(\alpha) \tag{7}$$

which takes care of the contribution originated from the coherent turbulent motion is related to the exponent ζ_3 of structure function through the relation

$$\langle \varepsilon_n \rangle \propto (\ell_n/\ell_0)^{\zeta_3 - 1}. \tag{8}$$

For the fully developed turbulence, the system is in the stationary state in which energy is transferring, hierarchically, from the largest eddies with diameter $\sim \ell_0$ to the smallest eddies with diameter about the order of the Kolmogorov length η with the rate $\langle \varepsilon_n \rangle$ independent of the diameter ℓ_n of eddies as described in the energy cascade model. In this case, $\zeta_3 = 1$. When turbulence is *not* fully developed, it does not reach its stationary state yet, and $\langle \varepsilon_n \rangle$ can be dependent on ℓ_n, i.e., $\zeta_3 \neq 1$.

The precise analyses of 4096^3 DNS [2] ($\varepsilon = 0.0831$ for $R_\lambda = 675$ and $\varepsilon = 0.0752$ for $R_\lambda = 1132$) by means of MPDFT showed that the turbulence with $R_\lambda = 675$ is in the fully developed turbulent state, but that the one with $R_\lambda = 1132$ is *not* fully developed yet. We obtained, self-consistently, the value of the exponent for $R_\lambda = 1132$ to be $\zeta_3 = 0.58$, i.e., the ℓ_n-dependence of $\langle \varepsilon_n \rangle$ turns out to be

$$\langle \varepsilon_n \rangle \propto (\ell_n/\eta)^{-0.42}. \tag{9}$$

The averaged energy dissipation rate $\langle \varepsilon_n \rangle$ associated with the coherent turbulent motion is smaller for larger eddies. It means that the turbulence with $R_\lambda = 1132$ does not have enough number of larger eddies to be a fully developed turbulence, i.e., the integral length may not reach its maximum value.

References

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