WHAT HAPPENS TO THE CRITICAL LAYER WHEN THE TRANSITION INDUCING MECHANISM IN THE SWIRLING FLOW IN AN ANNULUS SHIFTS FROM TOLLMIEN-SCHLICHTING'S TO TAYLOR'S?

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Abstract There are two distinct physical mechanisms that can induce transition in the class of fully developed swirling flows in an annulus, viz. the Tollmien-Schlichting and the Taylor mechanisms. The former is expected to dominate when the flow is maintained primarily by the imposed axial pressure gradient, and the latter to govern when it is rotation of the inner cylinder. There are several distinguishing physical features between these mechanisms, a salient one of which is the formation of a **critical layer** in the **Tollmien-Schlichting mechanism** and the **absence** of such a layer in the **Taylor mechanism**. The work submitted herewith for presentation at ETC 14 is an attempt to answer the question posed in the title. The study is done through investigation into characteristics of small-amplitude disturbance propagation conducted through application in parallel of both analytical and numerical techniques.

INTRODUCTION AND SCOPE OF THE WORK

The class of flows in question, which is the fully developed swirling flow in the annulus between concentric circular cylinders, is topologically characterised by helically wound streamlines. Transition is inducible in this class of flows by two distinct physical mechanisms, referred to in the following as the **Tollmien-Schlichting** and the **Taylor mechanism**s respectively. The decisive steering mechanism for transition in a basic flow of given swirl is its dynamic feature, which is whether it is maintained predominantly by the imposed axial pressure gradient (for the Tollmien-Schlichting mechanism) or by rotation of the inner cylinder about its own axis (for the Taylor mechanism). A measure of the relative importance of the two mechanisms is the swirl parameter, S_i, defined through $S_i = \frac{U_{ref\varphi}}{U_{refxp}}$ where, $U_{ref\varphi} = \Omega_i R_i$ and $U_{refxp} = -\frac{H^2}{2\mu}\frac{dP_G}{dx}$ with $2H = R_o - R_i$, in self-explanatory notation. An insight into the physics of the two steering mechanisms, gained by classical small-amplitude disturbance theory, is that in the **Tollmien-Schlichting mechanism**, disturbances causing transition propagate in the form of waves travelling at a certain speed, whereas in the **Taylor mechanism** disturbances are spatially stationary. Classical small-amplitude disturbance theory also reveals the existence of a **critical layer** associated with the travelling wave in the **Tollmien-Schlichting mechanism**, located at a postion at which the wave speed and the local flow speed are equal to each other. Since Reynolds stresses are primarily generated in the critical layer, see eg. [1], [2], an understanding of the mechanisms in the critical layer is crucial for treatment of any turbulent flow, swirling flows being no exception, hence the problem posed in this work. We address the question posed in the title by examining the solutions for the equations for propagation of small-amplitude disturbances in the class of swirling flows in question according to classical linear theory. Our work is to be seen against the background of work in relatively recent publications on this subject, [3], [4], [5], [6].

OUTLINE OF THE PRESENT WORK

The axial and azimuthal velocity components of the basic flow in our problem are those in the fully developed swirling flow in an annulus with a rotating inner cylinder. These may be written in a self-explanatory notation as: $V_{Gx} \simeq U_{refxp} U_{Gx}(y)$ and $V_{G\varphi} \simeq U_{ref\varphi} U_{G\varphi}(y)$. Our approach to illuminate the role of the critical layer in the problem is through examination of the equations for disturbance propagation in this basic flow from a perspective of its two limiting cases, which are $S_i \to 0$ and $S_i \to \infty$. The starting point for our investigation is the set of linearized equations for disturbances from which pressure has been eliminated by the two known standard procedures to yield the **Orr-Sommerfeld** and **Squire** equations respectively, and the **continuity equation**, see eg. [1], [2], [7]. This set may be non-dimensionalised in two ways, with the semi-gap width H as the reference length and either U_{refxp} or $U_{ref\varphi}$ as the reference velocity, according as $S_i \to 0$ or $S_i \to \infty$. We denote the Reynolds number using U_{refxp} as Re_{TS} and that using $U_{ref\varphi}$ as Re_T. The subscript TS and T have been used in that order to denote the dominance of the Tollmien-Schlichting or the Taylor mechanism respectively. We wish to draw the readers's/listeners' attention to the solutions of the disturbance sought in the well-known wave form,

$$(u_x, u_r, u_\varphi) = (A_x(y), A_r(y), A_\varphi(y)) \exp\left[i(\lambda_x x + n_\varphi \varphi - \omega t)\right] + c.c., \tag{1}$$

which leads to the **dispersion relation** of the problem that relates the frequency with the wavenumber and flow parameters. For axisymmetric disturbances this may be written symbolically as follows:

$$\omega = \omega(Re_{TS}, Re_T, \epsilon_R; \lambda_x), \tag{2}$$

wherein the frequency depends upon three parameters, $Re_{TS}, Re_T, \epsilon_R$, with $\epsilon_R = \frac{2H}{R_o+R_i}$. The occurrence of the critical layer in the flow is crucially dependent upon the **real part of the frequency** ω being **nonzero** , since only then does the disturbance propagate spatially at a non-zero velocity and the location of the critical layer determinable by setting this equal to the local flow velocity.

From the perspective of the limiting case of mild swirl for which we choose U_{refxp} as the reference velocity, the dispersion relation may be written as follows:

$$\omega_{TS} = \omega_{TS}(Re_{TS}, S_i \to 0, \epsilon_R; \lambda_x). \tag{3}$$

In the present state of algorithmic development, it is relatively straightforward to solve this eigenvalue problem numerically, see ([7]). This we have done for a range of parameters S_i and ϵ_R, both small, with MATLAB Programs along the lines of ([7]). The results bring out the dependence of the location of the critical layer on the transverse curvature ϵ_R, but **no dependence of the same on the swirl S_i is discernible**. A closer examination of the governing equations shows that this property is directly inferable from the structure of the equations themselves when formulated in the manner referred to.

From the perspective of the limiting case of strong swirl, when $U_{ref\varphi}$ is the appropriate reference velocity, the dispersion relation may formally be written in a form analogous to (3) as

$$\omega_T = \omega_T(Re_T, S_i \to \infty, \epsilon_R; \lambda_x), \tag{4}$$

a form which shows the dependence of ω_T on the three parameters, (Re_T, ϵ_R, S_i). However, this, in the authors' view, is not very well suited for the present purpose since, in the limit $S_i = \infty$ it leaves the relation to the solution of the classical Taylor's problem, and in particular to the role of the Taylor number as the decisive transition parameter, unclear. This relation is rendered transparent on rewriting the **Orr-Sommerfeld, Squire** and **Continuity** equations in **Taylor Variables**, $(\hat{t}, \hat{y}, \hat{\varphi}, \hat{x}; \hat{u}_r, \hat{u}_\varphi, \hat{u}_x)$ defined through:

$$\hat{t} = \frac{t}{Re_T}, \hat{y} = y, \hat{\varphi} = \varphi, \hat{x} = x; \hat{u}_r = u_r Re_T, \hat{u}_x = u_x Re_T, \hat{u}_\varphi = u_\varphi, \tag{5}$$

and subjecting the equations to the double limiting process $\epsilon_R \to 0, Re_T \to \infty$ with $Re_T^2 \epsilon_R = O(1)$. **The product $Re_T^2 \epsilon_R = O(1)$ may be identified with a Taylor number** Ta in which both characteristic lengths H and R_i enter. The physical meaning of the **Taylor variables** is evident from a closer inspection of (5), which may be seen to be that the time t is referred to $\frac{H^2}{\nu}$ instead of to $\frac{H}{\Omega_i R_i}$, the components of velocity disturbance (u_r, u_x) to $\frac{\nu}{H}$ instead of to $\Omega_i R_i$, and u_φ to $\Omega_i R_i$. A comparison of (5) with the original work of Taylor shows that this is the set of reference quantities used therein by Taylor. Noteworthy is the difference in the scales for (u_r, u_x) and u_φ.

The **modification of the critical layer on a shift of the transition inducing mechanism from Tollmien-Schlichting's to Taylor's** becomes lucid when the **Orr-Sommerfeld, Squire** and **Continuity** equations are written in **Taylor Variables**, recast in a form that brings out the analytical structure of the coefficient to exhibit the difference between the phase velocity of the wave and the flow velocity explicitly, and then examine the differential operators at location(s) where the phase velocity and the flow velocity are equal to each other. To this end the **Orr-Sommerfeld, Squire** and **Continuity** equations are written in a compact matrix notation which is as follows:

$$\mathbf{PA} + \left[\frac{\hat{\omega}}{\lambda_x} - \frac{\sqrt{Ta}}{S_i\sqrt{\epsilon_R}} U_{Gx}(y)\right] \mathbf{QA} = \mathbf{0}, \tag{6}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are matrix differential operators operating on the column vector $\mathbf{A} = (A_r, A_\varphi, A_x)^T$, see also (1). The equation (6) brings out the dependence of the location of the critical layer on the swirl parameter S_i. An analytical expression for $\hat{\omega}$ in the limit $S_i \to \infty$ has been derived. At the time of submitting this abstract to ETC 14, numerical values of this analytical expression are being evaluated.

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