AVAILABLE POTENTIAL ENERGY IN RAYLEIGH–BÉNARD CONVECTION

Graham Hughes[1], Bishakhdatta Gayen[1] & Ross Griffiths [1]
[1]*Research School of Earth Sciences, The Australian National University, Canberra ACT, Australia*

Abstract The energy budget for thermally-equilibrated Rayleigh–Bénard convection is developed theoretically, with explicit consideration of the role of available potential energy. The analysis shows that about half of the available potential energy generated by the thermal forcing is dissipated viscously by turbulence at high Rayleigh number. The remainder is consumed by diffusion acting to homogenise the temperature field. The results of direct numerical simulations of convection at Rayleigh numbers ranging from 10^8 to 10^{13} are also presented in support of this analysis. An important conclusion is that Rayleigh-Bénard convection may be viewed as a highly efficient mechanism of turbulent mixing in a stratified fluid.

THEORY

We describe the energetics of a volume of fluid heated from below and cooled from above. For simplicity the fluid is assumed Boussinesq with a linear equation of state, to which the generalised energetics framework shown schematically in Figure 1 may be applied (refer to [1] for a detailed derivation). The forms of mechanical energy considered are kinetic energy E_k and potential energy, decomposed into components of available potential energy E_a and background potential energy E_b (i.e. the potential energy that the fluid volume would have following an adiabatic relaxation of the density field to a state of no motion). The role of available potential energy appears to have been largely overlooked in previous energetics analyses of Rayleigh-Bénard convection.

The volume-integrated rates of conversion between the various forms of mechanical energy (Figure 1) are the buoyancy fluxes, $\overline{\Phi_z}$ and Φ'_z, viscous dissipation, $\bar{\epsilon}$ and ϵ', irreversible mixing of the density field, Φ_d, shear production, Φ_T, and working by boundary stresses, $\overline{\Phi_\tau}$ and Φ'_τ. The overbar and prime operators represent the decomposition of a given quantity into suitably defined 'mean' and 'fluctuating' components. The conversion rates Φ_{b2}, Φ_{b1} and Φ_i are associated with thermal forcing at the boundary. In physical terms Φ_{b2} represents the generation rate of available potential energy by maintaining a density field that is not in its adiabatically relaxed state, and Φ_{b1} gives the volume integrated rate of change of potential energy (and available potential energy) by net heating/cooling at each level. Φ_i corresponds to the rate at which the centre of mass of the volume would be raised or lowered by molecular diffusion if there was no convection. A number of energy conversions in Figure 1 can be in either direction, and an additional light grey arrow denotes the direction defined to be positive. For Rayleigh-Bénard convection, Φ_i and Φ_{b1} are expected to be negative and positive, respectively.

Attention is restricted here to thermally-equilibrated Rayleigh-Bénard convection, for which there is no net heating of the fluid volume. In this case, time-averaging and calculation of fluctuations about a time-average are closely related to the overbar and prime operators, respectively. In the thermally-equilibrated state, the amounts of mechanical energy present in a given form do not evolve systematically in time. Hence, consideration of the background potential energy in Figure 1 requires that $\Phi_d = \Phi_{b2}$. In physical terms, the tendency of irreversible mixing to homogenise the range of densities in the flow must be exactly offset by the action of thermal forcing to maintain that range. If working by boundary stresses is neglected, the kinetic energy budget requires that the total viscous dissipation $\bar{\epsilon} + \epsilon'$ balances the net buoyancy flux $\overline{\Phi_z} + \Phi'_z$. The available potential energy and internal energy budgets in turn require that the total viscous dissipation is also equal to both Φ_{b1} and Φ_i (an equivalent result was obtained by [2]).

The mathematical expressions for Φ_{b1} and Φ_{b2} [1] are used to show that these two conversion rates are equal, suggesting that Rayleigh-Bénard convection can be characterised by a mixing efficiency $\eta = \Phi_d/(\Phi_d + \bar{\epsilon} + \epsilon')$ that evaluates to 1/2 (when the working by boundary stresses is assumed negligible). This finding suggests that convective processes represent highly efficient mechanisms of mixing in a stratified fluid, a claim also supported in a recent study [3] showing that $\eta \to 1$ for horizontal convection at high Rayleigh number.

DIRECT NUMERICAL SIMULATIONS

The energy conversion rates in Figure 1 were calculated in a series of direct numerical simulations of Rayleigh-Bénard convection at Rayleigh numbers between 10^8 and 10^{13}. The simulations solve the continuity, momentum and temperature equations in three-dimensions using a mixed spectral/finite difference algorithm and a non-uniform grid in the vertical direction. The boundary conditions on the horizontal boundaries are no-slip and imposed temperature ($10\,^\circ$C and $40\,^\circ$C on the upper and lower boundaries, respectively). The domain is periodic in the horizontal directions and a variety of aspect ratios were examined. The working fluid is Boussinesq with properties corresponding to water nominally at $25\,^\circ$C. The results presented here are based on a period of simulation after a thermally-equilibrated state had been reached.

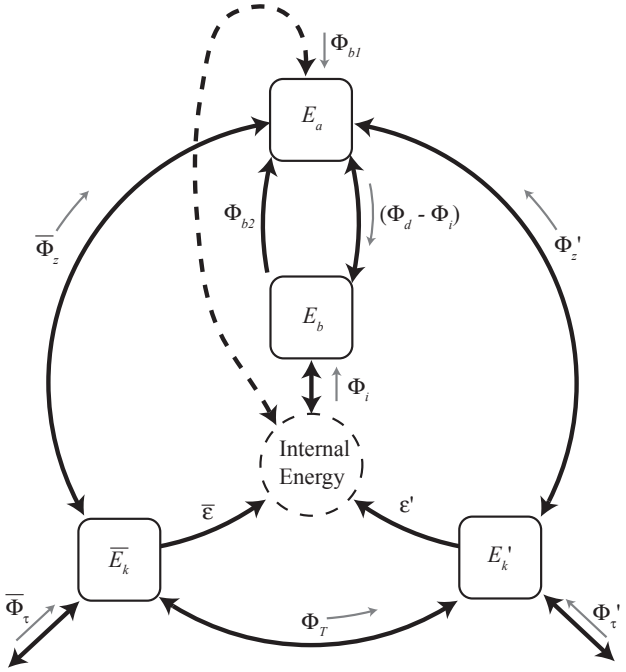

Figure 1. Schematic summary of the energetics framework for a volume of Boussinesq fluid with a linear equation of state (after [1]).

References

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