THE UNSTEADY FLOW WITHIN A ROTATING TORUS

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Abstract
We consider the temporal evolution of a viscous incompressible fluid in a torus of finite curvature; a problem first investigated by Madden and Mullin (1994), herein referred to as MM. The system is initially in a state of rigid-body rotation (about the axis of rotational symmetry) and the container's rotation rate is then changed impulsively. We describe the transient flow that is induced at small values of the Ekman number, over a time scale that is comparable to one complete rotation of the container. We show that (rotationally symmetric) eruptive singularities (of the boundary layer) occur at the inner or outer bend of the pipe for a decrease or an increase in rotation rate respectively. Moreover, there is a ratio of initial-to-final rotation frequencies for which eruptive singularities can occur at both the inner and outer bend simultaneously. We also demonstrate that the flow is susceptible to a combination of axisymmetric centrifugal and non-axisymmetric inflectional instabilities. The inflectional instability arises as a consequence of the developing eruption and is shown to be in qualitative agreement with the experimental observations of MM. Detailed quantitative comparisons are made between asymptotic predictions and finite (but small) Ekman number Navier–Stokes computations using a finite-element method.

INTRODUCTION

This work is directly motivated by the investigation of MM, which considered the internal flow induced by the sudden rotation of a toroidal container filled with an incompressible Newtonian fluid; the axis of rotation being the axis of rotational symmetry of the torus. Their work was largely experimental in nature, capturing the flow response by a combination of flow visualisation and LDV methods. In the initial stages an axisymmetric 'front' was observed at the outermost radius of the toroidal pipe. This front propagated radially inwards away from the wall, before propagating waves appeared in the around-torus direction, breaking the rotational symmetry of the flow. The rapid growth of these waves led ultimately to three-dimensional disordered flow (figure 19 of their paper presents a sequence of flow visualisation pictures that nicely captures the evolution, as obtained by a light sheet through the mid-plane of symmetry of the torus). MM speculate that the origin and break-up of the inwardly propagating front has two potential sources (pp. 241-242), a collisional boundary-layer phenomena and a centrifugal instability; however no detailed comparisons were possible to support either source.

Rather than restricting attention to the 'spin-up from rest' case considered by MM, we address a broader range of problems in which a transition is made from a rigid-body rotation at an initial frequency, Ω_i, to that at a second frequency, Ω_f, such that $|\Omega_i - \Omega_f| = O(1)$. This gives us a two-parameter problem (for a fixed torus curvature), with the flow governed by the rotation ratio $\Omega_r = \Omega_i/\Omega_f$ and an Ekman number $\mathrm{Ek} = \nu/(a^2\Omega_f)$, where a is the radius of the toroidal pipe.
Our aim is to clarify the physical origins of the fronts and waves observed in the experimental work of MM; full details of the physical origins of the fronts and waves can be found in [1].

MAIN RESULTS

We take a two-stranded approach to this problem: (i) we consider the $\mathrm{Ek} \ll 1$ limit in detail and describe the resulting unsteady boundary layer that develops in the torus together with its linear stability properties, (ii) we validate the conclusions of the boundary-layer results by large-scale Navier–Stokes computations at small, but finite, values of Ek.

We first determine the unsteady, axisymmetric base flow that develops subsequent to an impulsive change in the rotation rate of the container. We then consider the linear stability of this base flow to perturbations that are non-axisymmetric, by considering a Fourier component in the around-torus (axial) direction.

The underlying base flow in the $\mathrm{Ek} \ll 1$ limit is axisymmetric, unsteady, and evolves to a finite-time singularity at either the outermost point or/and innermost point of the torus (depending on Ω_r). The boundary-layer singularity is associated with a localised eruption into the bulk flow at finite but small Ek; good quantitative agreement is found between the unsteady boundary-layer solutions and the Navier–Stokes solutions. A key feature of the pre-eruption process is the introduction of inflexional velocity profiles in the dominant velocity component around the torus. We demonstrate the existence of an inviscid instability mechanism associated with these inflexional profiles that arises prior to the eruption of the boundary layer. By computating the unsteady evolution of a single Fourier mode (in the meridional cross section of the torus), in conjunction with the unsteady base flow, we demonstrate that the (local) asymptotic boundary-layer predictions are in agreement with the (global) finite-Ek response of the system.

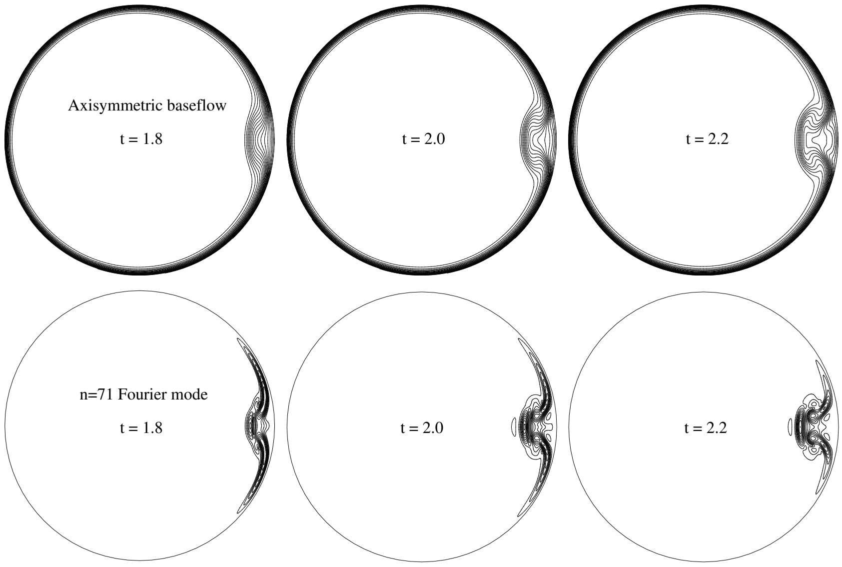

Figure 1: Contours of the axial (around torus) flow in the meridional cross section for Ek $= 1/2000$, $\Omega_r = 0$ (spin-up from rest) and the timescale shown is based on the final rotation frequency; the torus curvature is the value of the MM experimental configuration. Here we show the axisymmetric base flow and a non-axisymmetric Fourier mode (n being the around-torus wave number) of a linear disturbance. The disturbance becomes concentrated at the inflexion point induced by the eruption at the outermost point of the torus. In these cross sections, the axis of rotation is to the left of each image.

The results of our asymptotic analysis predict the peak growth rate is for perturbations with $n = k\varrho^{5/4}\text{Ek}^{-1/2}$ (where $k \approx 0.1$ to 0.11) waves around the torus. Here ϱ is a dimensionless measure of the distance of the eruption point from the axis of rotation (on the pipe-radius lengthscale) and k is a weakly varying function of the slower timescale over which the baseflow varies. Thus, the inviscid local analysis leads to an axial (around the torus) wave number of $n \approx 50$ for the fastest growing mode (taking $k \approx 0.105$) when Ek $= 1/1000$. Extrapolation of the section of torus shown in figure 19 of MM leads to the estimate that $n \approx 60$ in their experimental work at approximately the same value of Ek.

The 'fronts' observed in the experimental work of MM correspond to the collisional eruption of the centrifugally induced meridional flow within the boundary layer. The subsequent unstable waves can, we claim, be qualitatively linked to an inviscid instability of the near-eruption flow that grows on a faster timescale than that at which the baseflow develops, and hence can be described by a local eigenvalue problem.

The unsteady (transient) flow is susceptible to an axisymmetric centrifugal instability in the small time limit. In this case the growth of these modes is not governed by an eigenvalue problem because the timescale for the development of the disturbance is comparable to the timescale of the baseflow development. The development of these modes can therefore only be discussed in the context of an appropriate initial-value problem, based upon a linearised Navier–Stokes equation approach. We will demonstrate that, for the case of $\Omega_r > 0$, this instability occurs at the inner/outer-most radial position on the torus wall, whereas the inflexional modes develop at the outer/inner-most points. The two instability mechanisms therefore remain largely isolated from each other; this is not true if $\Omega_r < 0$ however.

Time permitting, we will present the results of experiments conducted in the fluid mechanics laboratory at the University of Auckland demonstrating many of the flow features discussed here.

References

[1] Hewitt, R., Hazel, A., Clarke, R. and Denier, J., Unsteady flow in a rotating torus after a sudden change in rotation rate, *Journal of Fluid Mechanics*, **688**, 88–119 (2011).

[2] Madden, F. N. and Mullin, T. The spin-up from rest of a fluid-filled torus. *Journal of Fluid Mechanics*, **265**, 217 (1994).