AN EXACT RELATION FOR COMPRESSIBLE MHD TURBULENCE

Supratik Banerjee[1] & Sébastien Galtier [1,2]
[1] *Institut d'Astrophysique Spatiale, Université Paris-Sud, Orsay, France*
[2]*Insitut Universitaire de France*

Abstract Compressible isothermal magnetohydrodynamic turbulence is discussed under the assumption of statistical homogeneity and in the asymptotic limit of large kinetic and magnetic Reynolds numbers. Following Kolmogorov, under the above context, we have derived an exact relation for some two-point correlation functions which generalizes our previous work for compressible hydrodynamics. We show that the magnetic field brings new source and flux terms into the dynamics. The introduction of a strong and uniform magnetic field simplifies significantly the exact relation for which a simple phenomenology is proposed. A prediction for axisymmetric energy spectra is also discussed. The principal domain of application of our theoretical work is the interstellar medium which is observed to be turbulent, magnetized and supersonic.

INTRODUCTION

Hydrodynamic turbulence, despite its ubiquitous nature, is extremely complex to be studied analytically. The degree of complexity gets considerably enhanced when the system consists of a magnetohydrodynamic (MHD) fluid. Yet several theoretical (analytical) works have been carried out in the framework of both the abovesaid cases when one assumes the incompressibility of the fluid. It was Kolmogorov who for the first time derived in 1941 an exact relation for the third-order moment of the velocity structure functions in incompressible hydrodynamic turbulence – the famous 4/5 law [5]. The corresponding exact relation was derived for incompressible MHD turbulence in 1998 [8] and its first generalization to compressible hydrodynamic turbulence has been realized only recently [4].

In this presentation, we shall discuss the extension of the compressible hydrodynamic case to compressible MHD [1]. The taking into consideration of compressibility of an MHD fluid leads us to get closer to the astrophysical reality taking place for example in the solar wind because (i) it is a plasma (so contains charged species), (ii) *in situ* data analysis has shown clear evidences of the effect of compressibility [2]. Moreover, turbulence inside interstellar clouds, being highly compressible (supersonic), also demands a theoretical background in order to be properly understood. A theoretical work in compressible MHD turbulence is also essential to analyze and understand the results of direct numerical simulations [7, 6, 3].

REDUCED FORM OF THE EXACT RELATION

Starting from the three-dimensional equations of isothermal compressible MHD, we can derive an exact relation under the assumptions of statistical homogeneity and high kinetic/magnetic Reynolds numbers [1]. The degree of complexity is however significantly higher in MHD than in hydrodynamics which renders the physical understanding more difficult. Hopefully, the exact relation gets considerably simplified if one supposes the existence of a strong external uniform magnetic field $\mathbf{B_0}$ (which corresponds to a sub-alfvénic turbulence regime). Additionally, if we neglect the fluctuations along the $\mathbf{B_0}$ direction, we obtain at main order [1]

$$-4\varepsilon = -\frac{B_0^2}{2}\left\langle (\nabla_\perp \cdot \mathbf{v}_\perp)\left(1+\sqrt{\frac{\rho}{\rho'}}\right) + (\nabla'_\perp \cdot \mathbf{v}'_\perp)\left(1+\sqrt{\frac{\rho'}{\rho}}\right)\right\rangle + B_0^2 \nabla_{\mathbf{r}_\perp} \cdot \left\langle \delta\left(\frac{1}{\sqrt{\rho}}\right)\delta(\sqrt{\rho})\delta\mathbf{v}_\perp \right\rangle , \quad (1)$$

where ε is the mean total energy injection rate (which is equal to the mean total energy dissipation rate), $\nabla_\perp$ implies derivatives transverse to $\mathbf{B_0}$, ρ is the density, $\mathbf{v}_\perp$ is the velocity component perpendicular to $\mathbf{B_0}$, $\mathbf{r}$ is the distance between the two points M, $M'$and $\delta X \equiv X' - X'$ is the increment. We may simplify the previous equation by assuming axisymmetry; the exact relation (1) can be written symbolically as (by using cylindrical coordinates)

$$-4\varepsilon = \mathcal{S}(r) + \frac{1}{r}\partial_r(r\mathcal{F}_r) , \quad (2)$$

where $\mathcal{S}$ is a source/sink term (first term in the right hand side of (1)) and $\mathcal{F}_r$ the radial component of the energy flux vector (last term in the right hand side of (1)). If we define an effective mean total energy injection rate as $\varepsilon_{\text{eff}} \equiv \varepsilon + \mathcal{S}/4$, a simple interpretation of expression (2) can be proposed as we see in Fig. 1: whereas for a direct cascade the energy flux vectors are oriented towards the axis of the cylinder, dilatation and compression are additional effects which act respectively in the opposite or in the same direction as the flux vectors (since terms like, $1+\sqrt{\frac{\rho'}{\rho}}$, are positive).

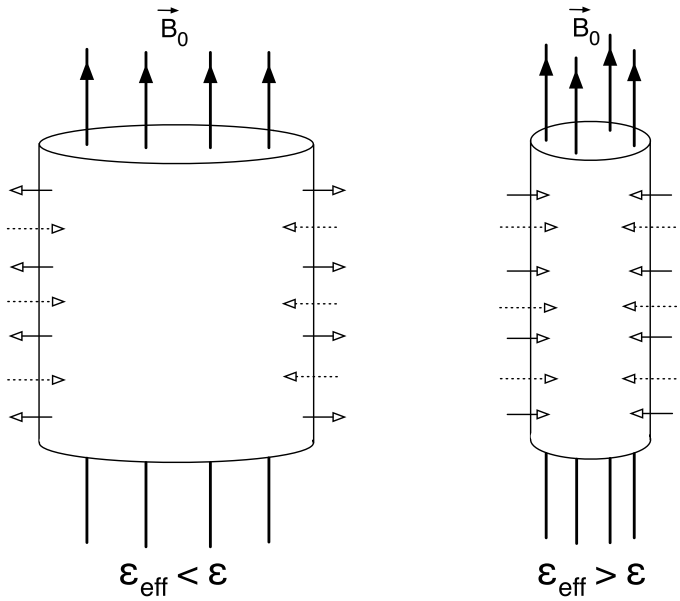

Figure 1. Dilatation (left) and compression (right) phases in space correlation for strongly magnetized MHD turbulence. In a direct cascade scenario the flux vectors (dotted arrows) are oriented towards the axis of the cylinder. Dilatation and compression (solid arrows) are additional effects which act respectively in the opposite or in the same direction as the flux vectors.

By the help of this reduced expression and dimensional analysis, we can set up a prediction for strongly magnetized compressible MHD turbulence; it is given by [1]

$$E^{\mathcal{B}}(k_\perp)\sqrt{E^{\mathcal{U}}(k_\perp)} \sim \varepsilon_{\rm eff} k_\perp^{-5/2}\,, \qquad (3)$$

where $\mathcal{B} \equiv \rho_\ell^{1/3} v_A$, $\mathcal{U} \equiv \rho_\ell^{1/3} v_\perp$, $\mathbf{v_A} \equiv \mathbf{b}/\sqrt{\mu_0\rho}$, with $\mathbf{b}$ the magnetic field. However, a power law steeper than $-5/2$ may be observed at large scales when compressible MHD turbulence becomes supersonic which corresponds to a $k_\perp$ dependence of $\varepsilon_{\rm eff}$.

CONCLUSION

The present work justifies its relevance by confirming theoretically the importance of the density-weighted variables in the scaling law of compressible MHD turbulence which has already been predicted numerically [6]. Moreover, we have been able to propose a simplified phenomenology for axisymmetric turbulence in the presence of a strong uniform external magnetic field. The interstellar and interplanetary media are certainly the best domains of application of our theory since observations show clearly the presence of turbulence which can be even supersonic in the interstellar clouds.

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