INSTANTON FILTERING FOR THE STOCHASTIC BURGERS EQUATION

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Abstract We address the question whether one can identify instantons in direct numerical simulations of the stochastically driven Burgers equation. For this purpose, we first solve the instanton equations using the Chernykh-Stepanov method [Phys. Rev. E **64**, (2001)]. These results are then compared to direct numerical simulations of the stochastic Burgers equation by extracting prescribed rare events from massive data sets of realizations. Using this approach we obtain the entire time history of the instanton evolution which allows us to identify the different phases predicted by the direct method of Chernykh and Stepanov with remarkable agreement. These results confirm the relevance of the instanton in turbulent flows.

Instantons for the stochastic Burgers equation

Understanding intermittency in turbulent flows is still one of the open problems in classical physics. More than 15 years ago, for certain systems like the problem of passive advection and Burgers turbulence the door for attacking this issue was opened by getting access to the probability density function to rare and strong fluctuations by the instanton approach [1–4]. Here, we concentrate on rare fluctuations in Burgers turbulence. For that case, Gurarie and Migdal [2] introduced the instanton approach and were able to calculate the instanton contribution to the right tail of the velocity increment probability distribution function (PDF). In succeeding work, Balkovsky et al. [4] were able to characterize the left tail of the increment PDF making use of the Cole-Hopf transformation [5, 6]. These analytical results were confirmed by direct numerical solution of the instanton equations by Chernykh and Stepanov [7].
The question remained unanswered whether one can observe or identify the instanton in numerical simulations of the stochastic Burgers equation. We present evindence that already at moderate Reynolds numbers the instanton can be identified in data sets of simulations of the stochastic Burgers equation. This gives a positive answer to this important question. In particular, we show by introducing a particular filtering technique that all phases of the instanton evolution can be recovered from data sets of simulations of the stochastic Burgers equation.
We consider the stochastically driven Burgers equation given by

$$u_t + uu_x - \nu u_{xx} = \phi \tag{1}$$

with a noise field ϕ that is δ-correlated in time and has finite correlation χ in space with correlation length L. Using the functional path integral introduced by Martin-Siggia-Rose/Janssen-de Dominicis [8–11], the PDF of the velocity gradients $u_x(t=0, x=0)$ can be written in terms of a path integral over the velocity field u and auxiliary field p, with a corresponding effective action. The saddle point (instanton) equations for the fields (u_I,p_I) yielding the largest contribution to the path integral for strong gradients are then given by

$$u_t + uu_x - \nu u_{xx} = -i \int \chi(x-x')p(x',t)dx' \tag{2a}$$

$$p_t + up_x + \nu p_{xx} = 4i\nu^2 \mathcal{F}\delta(t)\delta'(x) \ . \tag{2b}$$

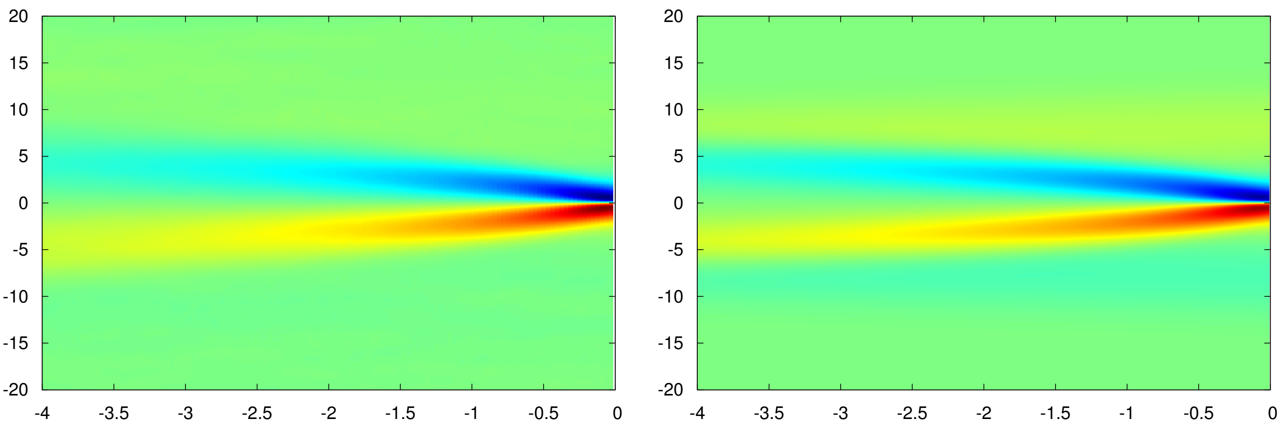

Figure 1. Comparison of the filtered velocity field (left) and the instanton field (right) as a space-time contour plot.

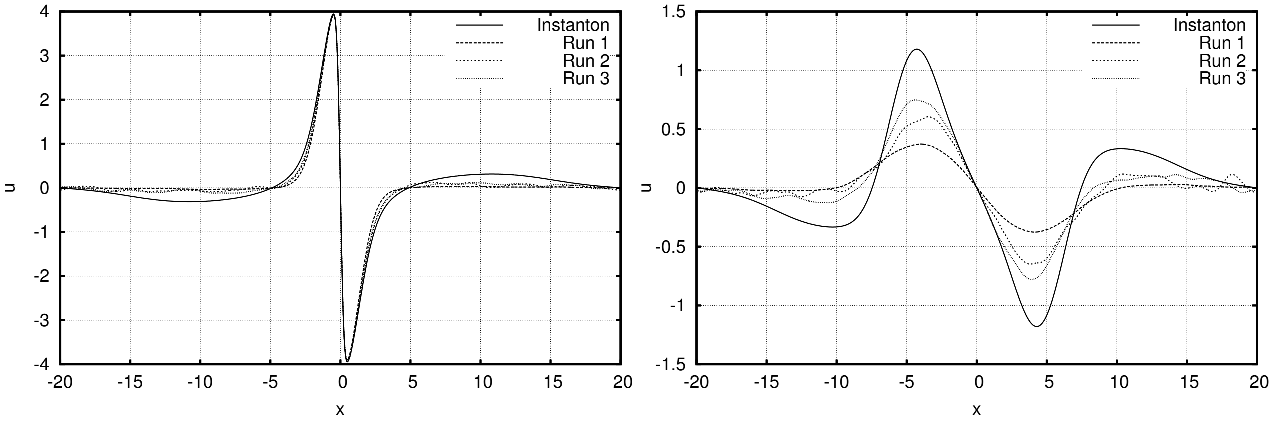

Figure 2. Comparison of the instanton field (solid) to stochastic simulations with varying hit percentages ($\approx 10\%$ (dashes), $\approx 0.5\%$ (dots), $\approx 0.05\%$ (small dots)) for $t = 0$ (left) and $t = -1.75$ (right). Agreement with the instanton approach increases with decreasing hit percentage.

These equations can be solved numerically using an iterative algorithm proposed [7]. We also note the similarity of the system (2) to equations that arise in the context of transition probabilities [12, 13].

Extracting the instanton

In order to provide a sufficient data set for the extraction of the instanton from simulations of the stochastic Burgers equation (1), we conducted the following numerical experiment: We started the integration of the stochastic Burgers equation from zero initial conditions for the velocity field $u(t = t_{\text{min}}, x) = 0$ at a large negative time t_{min} up to the final time $t = 0$. This single experiment was repeated $\approx 10^7$ times using the 64 CUDA Tesla 1060 graphics on the Bochum GPU Cluster and the 96 CUDA Fermi 2050 graphics cards on the CUNY GPU Cluster. The total simulation length obtained by this parallelism corresponds to $\approx 10^8$ integral times. In this data-set, we scanned for events with a prescribed velocity gradient and averaged over them, after shifting the event to the origin. We thus obtain an ensemble average for the velocity $\langle u(t, x)\rangle$ and the force $\langle \phi(t, x)\rangle$ in space and time. Thus, for sufficiently strong velocity gradients $u_x(0, 0) = a$, the averaged solutions $\langle u(t, x)\rangle$ and $\langle \phi(t, x)\rangle$ supposedly coincides with the instanton solution of (2). Fig. (1) shows the filtered field $\langle u(t, x)\rangle$ (left) and the instanton field $u_I(t, x)$ (right). Although the filtered field shows a slightly shorter extent in time, the congruence is clearly visible. The rareness of the filtered events has a strong impact on the agreement between the instanton approximation and the full stochastic simulation. In order to demonstrate the varying resemblance to the instanton approximation, we alter the probability of reaching a prescribed velocity gradient by changing the kinematic viscosity ν. Fig. (2) shows the filtered field $\langle u(t, x)\rangle$ and the instanton field $u_I(t, x)$ for three different hit percentages. As the rareness of the event increases, accordance with the instanton grows considerably. Especially the velocity gradient in the origin is only reproduced when the events are rare.

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