NEW TURBULENT SCALING LAWS FROM THE MULTI-POINT CORRELATION EQUATIONS

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Abstract In order to describe statistical quantities of turbulent flows, it is our aim to deduce scaling laws from the multi-point correlation equations. The mathematical method employed will be the Lie-point symmetries. The method is rather generic and will be applied to different flows, such as channel flows with and without rotation.

EQUATIONS OF STATISTICAL TURBULENCE THEORY

The velocity $\boldsymbol{U}$ and the normalized pressure P are decomposed according to the Reynolds decomposition, i.e. $\boldsymbol{U} = \bar{\boldsymbol{U}} + \boldsymbol{u}$ and $P = \bar{P} + p$, where the overbar denotes averaged quantities and fluctuations are given by lower case letters. With this the Reynolds averaged Navier-Stokes equations write

$$\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_k \frac{\partial \bar{U}_i}{\partial x_k} = -\frac{\partial \bar{P}}{\partial x_i} + \nu \frac{\partial^2 \bar{U}_i}{\partial x_k \partial x_k} - \frac{\partial \overline{u_i u_k}}{\partial x_k}\,,\ i = 1, 2, 3\,,$$

where $t \in \mathbb{R}^+$ and $\boldsymbol{x} \in \mathbb{R}^3$ represent time and position vector. The viscosity ν is a positive constant.
We introduce the multi-point approach [1] to deal with the closure problem, represented through the Reynolds stress tensor $\overline{u_i u_k}$. Considering the infinite set of correlation equations has the advantage that the closure problem is somehow bypassed. Furthermore, the multi-point correlation delivers additional information on the turbulence statistics such as length scale information which may not be gained from the Reynolds stress tensor alone.
A multi-point correlation (MPC) shall be given by a multiplication of fluctuation velocities and its average, represented as $R_{i_{\{n+1\}}} = R_{i_{(0)} i_{(1)} \ldots i_{(n)}} = \overline{u_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot u_{i_{(n)}}(\boldsymbol{x}_{(n)})}$. From the Navier-Stokes equations follows the transport equation of the MPC

$$\begin{aligned}\mathcal{T}_{i_{\{n+1\}}} = \frac{\partial R_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \Bigg[& \bar{U}_{k_{(l)}}(\boldsymbol{x}_{(l)}) \frac{\partial R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}} + R_{i_{\{n+1\}}[i_{(l)} \mapsto k_{(l)}]} \frac{\partial \bar{U}_{i_{(l)}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} + \frac{\partial P_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} \\ & - \nu \frac{\partial^2 R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} - R_{i_{\{n\}}[i_{(l)} \mapsto \emptyset]} \frac{\partial \overline{u_{i_{(l)}} u_{k_{(l)}}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} + \frac{\partial R_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}} \Bigg] = 0 \end{aligned} \qquad (1)$$

for $n = 1, \ldots, \infty$. The first tensor equation of this infinite chain propagates $R_{i_{\{2\}}}$ which has a close link to the Reynolds stress tensor, i.e. $\lim_{x_{(k)} \to x_{(l)}} R_{i_{\{2\}}} = \lim_{x_{(k)} \to x_{(l)}} R_{i_{(0)} i_{(1)}} = \overline{u_{i_{(0)}} u_{i_{(1)}}}(\boldsymbol{x}_{(l)})$ with $k \neq l$. These equations have to be completed by continuity equations and further permutation conditions, such as $R_{ij}(\boldsymbol{x}_{(0)}, \boldsymbol{x}_{(1)}) = R_{ji}(\boldsymbol{x}_{(1)}, \boldsymbol{x}_{(0)})$.

LIE POINT SYMMETRIES

The Lie-point analysis allows us to derive special invariant solutions of partial differential equations, which, as will be seen later, verify known and new scaling laws of turbulent flows. The first step is to find Lie-point symmetries of the given PDE, in our case of the MPC equations (1). These symmetries are transformations of the independent variables $t, \boldsymbol{x}_{(0)}, \boldsymbol{x}_{(1)}, \ldots$ and the dependent functions $\bar{U}, \bar{P}, R_{i_{\{n\}}}, P_{i_{\{n-1\}}[q]}$, where the transformed equations are equivalent to the multi-point equations, i.e. form invariant under these transformations.
As expected, all symmetries of the Navier-Stokes equations can be transferred to the multi-point equations. The further remarkable result is that additional symmetries arise, which have no direct counterparts in the Navier-Stokes equations [4]. One of these additional symmetries, a scaling symmetry

$$\bar{T}_s:\ t^* = t,\ \ \boldsymbol{x}^* = \boldsymbol{x},\ \ \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)},\ \ \bar{U}^*_{i_{(0)}} = \mathrm{e}^{a_s} \bar{U}_{i_{(0)}},\ \ R^*_{i_{(0)} i_{(1)}} = \mathrm{e}^{a_s} \left[R_{i_{(0)} i_{(1)}} + (1 - \mathrm{e}^{a_s}) \bar{U}_{i_{(0)}} \bar{U}_{i_{(1)}} \right],\ \ \cdots,$$

plays an important role when calculating scaling laws to be shown below.

TURBULENT SCALING LAWS

Using group theoretical methods the symmetries of the MPC equations are employed to construct so-called invariant solutions of the MPC equations, which will represent scaling laws. Various scaling laws for different turbulent flows will subsequently be derived.

In the case of decaying homogeneous isotropic turbulence various different classes of scaling laws have been derived [4]. Next to the classical solution, where the kinetic energy decays algebraically with $k \sim (t + t_0)^m$ also a exponential decay $k \sim e^{-t/t_0}$ can be found in order to describe decaying turbulence generated by a fractal grid.

As a second case, we consider wall-bounded turbulent shear flows, where we focus on the logarithmic law of the wall for whose calculation the new scaling symmetry is essential. For the reason of briefness we skip the lengthy computations and obtain the invariance condition for the MPC equation. Since x_2^+ is the wall-normal direction, the log-law can be found in the classical dimensionless form

$$u^+ = \frac{1}{\kappa} \ln(x_2^+ + A^+) + C \ . \tag{2}$$

The method of Lie symmetries is not restricted to mean velocities, so that two-point correlations and consequently also the Reynolds stresses can be calculated. Compared to the results in [4], presently new symmetries were found. The stresses have a more complex form and we expect an improved scaling law that will allow us to fit the data more precisely.

As a third test case a rotating turbulent channel flow is considered, for which different scaling laws are calculated depending on the direction of the rotational axis $\mathbf{\Omega}$.

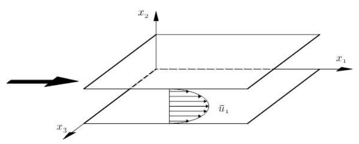

Figure 1. Flow geometry of the pressure driven channel flow.

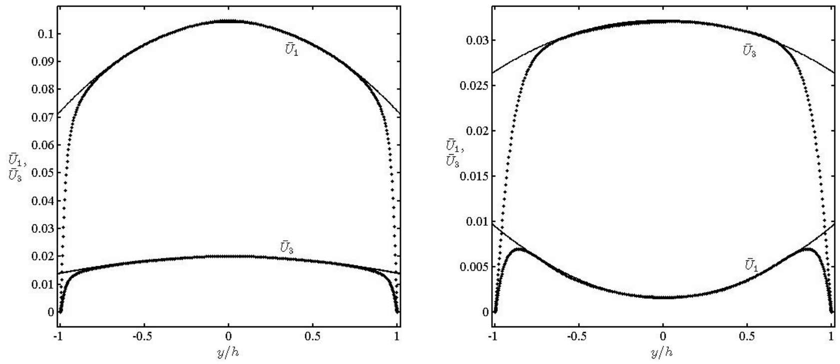

Figure 2. Comparison of the scaling law $(-)$ in (3) with the DNS data $(\cdots)$ of [2] at $Ro_2 = 0.011$(left) and $Ro_2 = 0.18$ (right).

First, we assume that the rotational axis lies along the x_3 direction i.e. only Ω_3 is non-zero. Applying Lie symmetry analysis, the result for the averaged velocity is $\bar{U}_1(x_2) = Cx_2^\beta + A$, where C, β and A are constants. DNS and experiments suggest $\beta = 1$ and from a re-scaling based on Ω_3 we obtain the well-known scaling law for a rotating channel about the x_3-axis $\bar{U}_1(x_2) = \alpha_{rot}\Omega_3 x_2 + \bar{U}_{cl}$ (see [3]). Comparing these scaling laws to the DNS of [5] a clear validation can be found, where the value for α_{rot} appears to be very close to 2.

Next, assuming rotation about x_2, two velocity components $\bar{U}_1$ and $\bar{U}_3$ have to be considered since the Coriolis force induces a cross flow. Rewriting the underlying symmetries in a rotating frame the resulting scaling laws are of the form:

$$\begin{aligned} \bar{U}_1 &= \left(\frac{x_2}{h}\right)^b \left[a_1 \cos\left(cRo_2 \cdot \ln \frac{x_2}{h}\right) + a_2 \sin\left(cRo_2 \cdot \ln \frac{x_2}{h}\right)\right] + d_1(Ro_2) \\ \bar{U}_3 &= \left(\frac{x_2}{h}\right)^b \left[a_1 \sin\left(cRo_2 \cdot \ln \frac{y}{h}\right) - a_2 \cos\left(cRo_2 \cdot \ln \frac{x_2}{h}\right)\right] + d_2(Ro_2) \, . \end{aligned} \tag{3}$$

We compare the DNS data of [2] at $Re_\tau = 360$ with the scaling law (3) in the Figure 2. Two different rotation numbers $Ro_2 = \frac{2\Omega_2 h}{u_{\tau 0}}$ are considered while an excellent fit in the center of the channel can be determined for all cases. Here, $u_{\tau 0}$ refers to the friction velocity of the non-rotating case. The DNS data in [2] provide that with an increasing Ω_2 the magnitude of $\bar{U}_1$ and $\bar{U}_3$ switch positions since with increasing rotation rates $\bar{U}_1$ is suppressed while $\bar{U}_3$ increases up to a certain point and decreases again though to a smaller extend compared to $\bar{U}_1$.

Currently, the method of Lie symmetries is applied to several other flows such as decaying turbulence in a rotating frame and we expect to obtain various new scaling laws.

References

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